

Energy of short quantum strings in $AdS_5 \times S^5$

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[R. Roiban, AT](#), [arXiv:0906.4294](#), [arXiv:1102.1209](#)

[M. Beccaria, S. Giombi, G. Macorini, R. Roiban, AT](#), [arXiv:1203.5710](#)

Closely related work:

[N. Gromov, D. Serban, I. Shenderovich, D. Volin](#), [arXiv:1102.1040](#)

[B. Basso](#), [arXiv:1109.3154](#)

[N. Gromov and S. Valatka](#), [arXiv:1109.6305](#)

Maximally symmetric case of gauge-string duality:

[Maldacena, Polyakov, Klebanov, Witten,...]

planar $\mathcal{N} = 4$ super Yang-Mills \leftrightarrow free $AdS_5 \times S^5$ superstring
closed string states on $R \times S^1 \leftrightarrow$ gauge-inv. SYM states on $R \times S^3$
marginal str. vertex ops on $R^2 \leftrightarrow$ conf. primary SYM ops on R^4

correlators of $AdS_5 \times S^5$ string vertex operators

– analogs of S-matrix elements in flat 10d space —

are (expected to be) dual to correlators of

conformal operators of planar $\mathcal{N} = 4$ SYM

In particular, relation of 2-point functions means that

spectrum of $AdS_5 \times S^5$ string energies

\leftrightarrow spectrum of dimensions of SYM primary operators

Then the spectrum of $\mathcal{N} = 4$ SYM conformal dimensions $\Delta(\lambda)$

should be possible to describe by 2d $AdS_5 \times S^5$ superstring sigma

Integrability:

allows “in principle” to solve the problem of **spectrum**
enormous progress in the last 10 years

Some key inputs:

- SYM action + perturbation theory ($\lambda \ll 1$)
- $AdS_5 \times S^5$ GS superstring action + α' -expansion ($\sqrt{\lambda} \gg 1$)
- classical integrability of $AdS_5 \times S^5$ GS action
- perturbative integrability of SYM spectral problem:
(1-loop, 2-loop, ...) dilatation operator = spin chain Hamiltonian
[Minahan, Zarembo; Beisert, Staudacher, ...]
- guidance from large-charge limits: BMN, GKP, FT

Assume **integrability** extends to all orders on both sides

- construct interpolating Bethe ansatz guided by general principles, symmetries and data from both weak+strong coupling
- check consistency of its predictions

I. Spectrum of “long” operators / “semiclassical” string states

determined by **Asymptotic Bethe Ansatz** (2002-2007)

- its final [Beisert-Eden-Staudacher] form found by intricate superposition of data from $\lambda \ll 1$ gauge theory (spin chain, BA,...) and perturbative string theory (classical and 1-loop phase, BMN), symmetries (S-matrix), assumption of exact integrability

- consequences **checked** against available gauge and string data

Key example: **cusp anomalous dimension** – of $\text{Tr}(\Phi D^S \Phi)$

$$\Delta = S + 2 + f(\lambda) \ln S + \dots, \quad S \gg 1$$

$$f_{\lambda \ll 1} = \frac{\lambda}{2\pi^2} \left[1 - \frac{\lambda}{48} + \frac{11\lambda^2}{45 \cdot 2^8} - \left(\frac{73}{630} + \frac{4\zeta_3^2}{\pi^6} \right) \frac{\lambda^3}{2^7} + \dots \right]$$

$$f_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi} \left[1 - \frac{3 \ln 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} - \dots \right]$$

$$\zeta_k = \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}, \quad K = \beta(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915\dots$$

from 2-loop string sigma-model integrals [Roiban, Tirziu, AT]

exact integral eq. [Basso, Korchemsky, Kotanski]: any order term

II. Spectrum of “short” operators = quantum string states

Thermodynamic Bethe Ansatz (2005-...)

- reconstructed from ABA using solely methods/intuition of 2-d integrable QFT, i.e. inspired by string-theory side
- highly non-trivial construction – lack of 2-d Lorentz invariance in standard BMN-vacuum-adapted l.c. gauge
- in few cases ABA “improved” by Luscher corrections is enough: 4- and 5-loop Konishi dim, 4-loop dim. of twist 2 operator
- complicated set of integral equations in need of simplification; so far predictions extracted only numerically starting from weak coupling and interpolating to larger λ
- need more data to **check predictions** at $\lambda \ll 1$ and $\lambda \gg 1$, i.e. against perturbative gauge-theory and string-theory data

Key example:

dimension $\Delta = 2 + \gamma(\lambda)$ of **Konishi operator** $\text{Tr}(\bar{\Phi}_i \Phi_i)$

$$\begin{aligned} \gamma(\lambda \ll 1) = & \frac{12\lambda}{(4\pi)^2} \left[1 - \frac{4\lambda}{(4\pi)^2} + \frac{28\lambda^2}{(4\pi)^4} \right. \\ & - (208 - 48\zeta_3 + 120\zeta_5) \frac{\lambda^3}{(4\pi)^6} \\ & \left. + 8(158 + 72\zeta_3 - 54\zeta_3^2 - 90\zeta_5 + 315\zeta_7) \frac{\lambda^4}{(4\pi)^8} + \dots \right] \end{aligned}$$

5-loop result first found using integrability [Banjok,Janik]

confirmed by more standard methods [Velizhanin; Eden et al 12]

Suppose one can sum up (convergent) $\lambda \ll 1$ expansion
and then re-expand at $\lambda \gg 1$

What one should expect to get for $\gamma(\lambda \gg 1)$?

Duality to [string theory](#) predicts the structure

of strong-coupling expansion:

leading term – near-flat-space expansion for fixed quant. numbers

[Gubser, Klebanov, Polyakov 98]

$$\Delta = \sqrt{2N\sqrt{\lambda}} + \dots$$

Subleading terms: $\alpha' = \frac{1}{\sqrt{\lambda}}$ expansion of 2d anom. dimensions

of corresponding vertex operators [Roiban, AT 09] ($N = 2$)

$$\begin{aligned} \gamma(\lambda \gg 1) &= 2\sqrt[4]{\lambda} + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{(\sqrt[4]{\lambda})^3} + \frac{b_3}{(\sqrt[4]{\lambda})^{5/2}} + \dots \\ &= 2\sqrt[4]{\lambda} \left[1 + \frac{b_1}{2\sqrt{\lambda}} + \frac{b_2}{2(\sqrt{\lambda})^2} + \frac{b_3}{2(\sqrt{\lambda})^3} + \dots \right] \end{aligned}$$

Values of b_k from string theory? From TBA?

Dimensions of “short” SYM operators

= energies of quantum string states

find leading $\alpha' = \frac{1}{\sqrt{\lambda}}$ corrections to energy of
“lightest” massive string states on first massive string level
dual to operators in **Konishi multiplet** in SYM theory
– compare with predictions of TBA approach

important to check integrability-based approach
which involves subtle assumptions
directly against perturbative string sigma model

TBA results:

start at weak coupling for $sl(2)$ Konishi descendant $\text{Tr}(\Phi D^2 \Phi)$

use TBA to find $\Delta(\lambda)$ numerically;

match to expected form of strong-coupling expansion to extract b_k

[Gromov, Kazakov, Vieira 09; Frolov 10, 12]

$$b_1 \approx 1.988, \quad b_2 \approx -3.07$$

Compare to string theory:

One can find b_k using **semiclassical “short string”** expansion

[Roiban, AT 09, 11; Gromov, Serban, Shenderovich, Volin 11]

$$b_1 = 2, \quad b_2 = a - 3\zeta_3$$

rational a was found [Gromov, Valatka 11] using “2-loop”

coefficient in exact slope function ($E^2 = h(\lambda)S$) [Basso 11]

$$b_2 = \frac{1}{2} - 3\zeta_3 \approx -3.106\dots$$

Remarkable agreement with TBA - check of quantum integrability

Konishi state

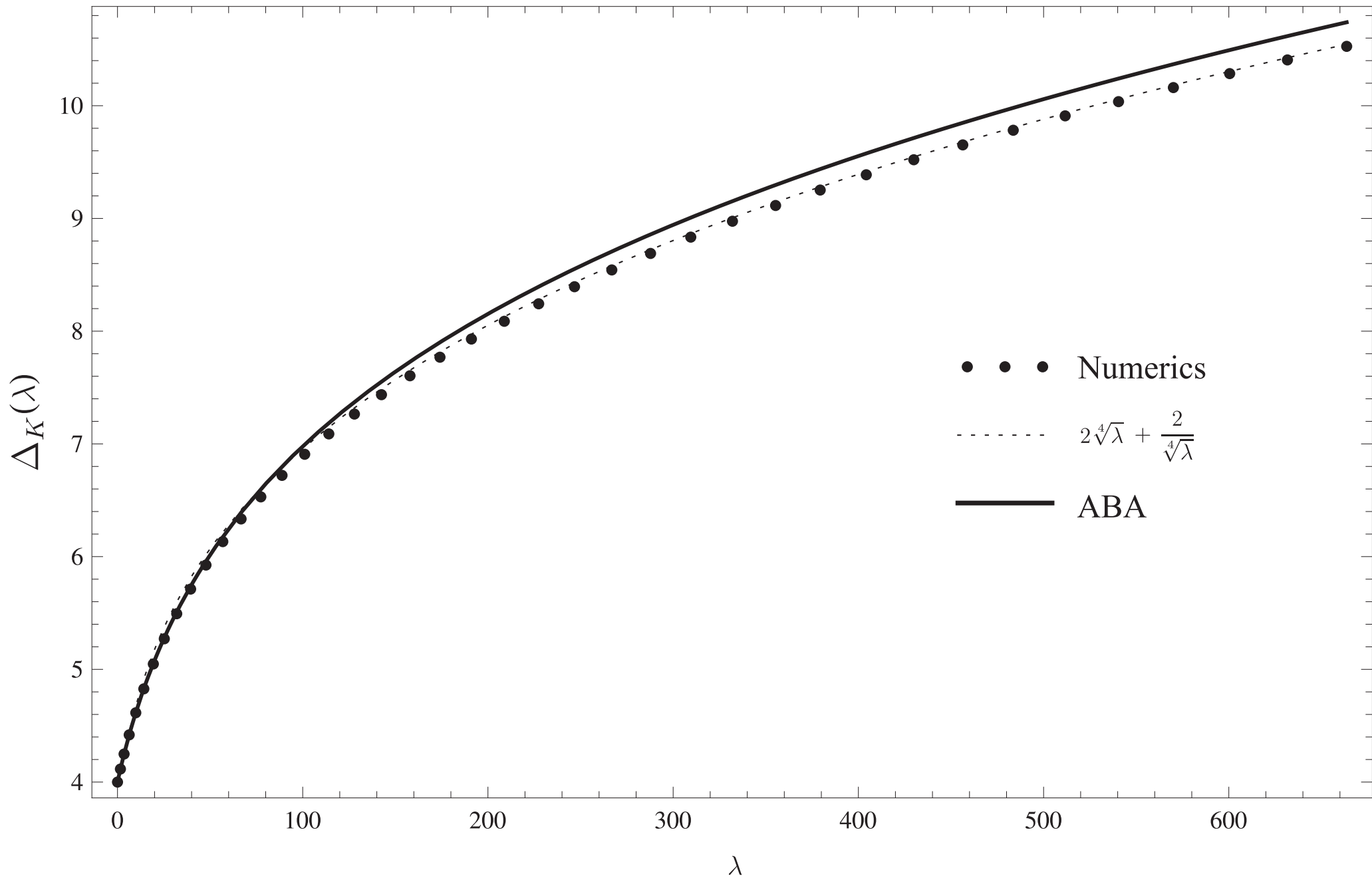


Figure 1: Plot from Gromov, Kazakov, Vieira [09]

Recent work on string side: [BGMRT 12]

- **highest transcendentality** terms in b_k are $\sim \zeta_{2k-1}$ and have 1-loop origin, e.g.,

$$b_3 = a_1 + a_2 \zeta_3 + a_3 \zeta_5, \quad a_3 = \frac{15}{2}$$

rational a_1 receives contribution from 3 loops; a_2 from 2-loops, etc.; $b_4 \sim \zeta_7 + \dots$, etc.

- **supermultiplet structure**: universality of coefficients in E for string states with spins in different $AdS_5 \times S^5$ directions: dual operators from Konishi multiplet have same energy (up to constant shift depending on position in the multiplet)
- **states on leading Regge trajectory**: general structure of dependence of energy on string tension $\sqrt{\lambda}$, string level (spin) and S^5 orbital momentum J

Some open questions:

- Analytic form of strong-coupling expansion from TBA?
- only ζ_k coefficients in $\Delta(\lambda)$ in both weak and strong coupling expansions or other transcendental constants may also appear? (cf. cusp anomalous dimension)
2-loop string computation may shed light on this ...
- Asymptotic form of strong coupling expansion:
 $e^{-k\sqrt{\lambda}}$ corrections to cusp dimension
absent for short strings / operators like Konishi?
[no such corrections in leading slope; no massless S^5 modes,...]
- Energies of other quantum states: similar behavior?

Konishi multiplet:

long multiplet related to singlet $[0, 0, 0]_{(0,0)}$ by susy

$$[J_2 - J_3, J_1 - J_2, J_2 + J_3]_{(s_L, s_R)}$$

$$s_{L,R} = \frac{1}{2}(S_1 \pm S_2)$$

$SO(6)$ (J_1, J_2, J_3) and $SO(4)$ (S_1, S_2) labels
of $SO(2, 4) \times SO(6)$ global symmetry

[Andreanopoli, Ferrara 98; Bianchi, Morales, Samtleben 03]

$$\Delta = \Delta_0 + \gamma(\lambda), \quad \Delta_0 = 2, \frac{5}{2}, 3, \dots, 10$$

same anomalous dimension γ for all members

singlet eigen-state of anom. dim. matrix with **lowest** eigenvalue

Examples of gauge-theory operators in Konishi multiplet:

$[0, 0, 0]_{(0,0)}$:

$$\text{Tr}(\bar{\Phi}_i \Phi_i), \quad i = 1, 2, 3, \quad \Delta_0 = 2$$

$[2, 0, 2]_{(0,0)}$:

$$\text{Tr}([\Phi_1, \Phi_2]^2) \text{ in } su(2) \text{ sector}, \quad \Delta_0 = 4$$

$[0, 2, 0]_{(1,1)}$:

$$\text{Tr}(\Phi_1 D^2 \Phi_1) \text{ in } sl(2) \text{ sector}, \quad \Delta_0 = 4$$

Δ_0	
2	$[0, 0, 0]_{(0,0)}$
$\frac{5}{2}$	$[0, 0, 1]_{(0, \frac{1}{2})} + [1, 0, 0]_{(\frac{1}{2}, 0)}$
3	$[0, 0, 0]_{(\frac{1}{2}, \frac{1}{2})} + [0, 0, 2]_{(0,0)} + [0, 1, 0]_{(0,1)+(1,0)} + [1, 0, 1]_{(\frac{1}{2}, \frac{1}{2})} + [2, 0, 0]_{(0,0)}$
$\frac{7}{2}$	$[0, 0, 1]_{(\frac{1}{2}, 0)+(\frac{1}{2}, 1)+(\frac{3}{2}, 0)} + [0, 1, 1]_{(0, \frac{1}{2})+(1, \frac{1}{2})} + [1, 0, 0]_{(0, \frac{1}{2})+(0, \frac{3}{2})+(1, \frac{1}{2})} + [1, 0, 2]_{(\frac{1}{2}, 0)}$ $+ [1, 1, 0]_{(\frac{1}{2}, 0)+(\frac{1}{2}, 1)} + [2, 0, 1]_{(0, \frac{1}{2})}$
4	$[0, 0, 0]_{(0,0)+(0,2)+(1,1)+(2,0)} + [0, 0, 2]_{(\frac{1}{2}, \frac{1}{2})+(\frac{3}{2}, \frac{1}{2})} + [0, 1, 0]_{2(\frac{1}{2}, \frac{1}{2})+(\frac{1}{2}, \frac{3}{2})+(\frac{3}{2}, \frac{1}{2})} + [2, 0, 2]_{(0,0)}$ $+ [0, 1, 2]_{(1,0)} + [0, 2, 0]_{2(0,0)+(1,1)} + [1, 0, 1]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [1, 1, 1]_{2(\frac{1}{2}, \frac{1}{2})} + [2, 0, 0]_{(\frac{1}{2}, \frac{1}{2})}$
6	$[0, 0, 0]_{3(0,0)+3(1,1)+(2,2)} + [0, 0, 2]_{3(\frac{1}{2}, \frac{1}{2})+(\frac{1}{2}, \frac{3}{2})+(\frac{3}{2}, \frac{1}{2})+(\frac{3}{2}, \frac{3}{2})} + [0, 1, 0]_{4(\frac{1}{2}, \frac{1}{2})+2(\frac{1}{2}, \frac{3}{2})+2(\frac{3}{2}, \frac{1}{2})+}$ $+ [0, 1, 2]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [0, 2, 0]_{3(0,0)+(0,1)+(0,2)+(1,0)+3(1,1)+(2,0)} + [0, 2, 2]_{(\frac{1}{2}, \frac{1}{2})}$ $+ [0, 3, 0]_{2(\frac{1}{2}, \frac{1}{2})} + [0, 4, 0]_{(0,0)} + [1, 0, 1]_{(0,0)+3(0,1)+3(1,0)+4(1,1)+(1,2)+(2,1)} + [1, 0, 3]_{(\frac{1}{2}, \frac{1}{2})} + [0, 1, 3]_{(\frac{1}{2}, \frac{1}{2})}$ $+ [1, 1, 1]_{4(\frac{1}{2}, \frac{1}{2})+2(\frac{1}{2}, \frac{3}{2})+2(\frac{3}{2}, \frac{1}{2})} + [1, 2, 1]_{(0,0)+(0,1)+(1,0)} + [2, 0, 0]_{3(\frac{1}{2}, \frac{1}{2})+(\frac{1}{2}, \frac{3}{2})+(\frac{3}{2}, \frac{1}{2})+(\frac{3}{2}, \frac{3}{2})}$ $+ [2, 0, 2]_{(0,0)+(1,1)} + [2, 1, 0]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [2, 2, 0]_{(\frac{1}{2}, \frac{1}{2})} + [3, 0, 1]_{(\frac{1}{2}, \frac{1}{2})} + [4, 0, 0]_{(0,0)}$
$\frac{17}{2}$	$[0, 0, 1]_{(0, \frac{1}{2})+(0, \frac{3}{2})+(1, \frac{1}{2})} + [0, 1, 1]_{(\frac{1}{2}, 0)+(\frac{1}{2}, 1)} + [1, 0, 0]_{(\frac{1}{2}, 0)+(\frac{1}{2}, 1)+(\frac{3}{2}, 0)} + [1, 0, 2]_{(0, \frac{1}{2})}$ $+ [1, 1, 0]_{(0, \frac{1}{2})+(1, \frac{1}{2})} + [2, 0, 1]_{(\frac{1}{2}, 0)}$
9	$[0, 0, 0]_{(\frac{1}{2}, \frac{1}{2})} + [0, 0, 2]_{(0,0)} + [0, 1, 0]_{(0,1)+(1,0)} + [1, 0, 1]_{(\frac{1}{2}, \frac{1}{2})} + [2, 0, 0]_{(0,0)}$
$\frac{19}{2}$	$[0, 0, 1]_{(\frac{1}{2}, 0)} + [1, 0, 0]_{(0, \frac{1}{2})}$
10	$[0, 0, 0]_{(0,0)}$

Table 1: Long Konishi multiplet (a bit cut off)

Comparison between gauge and string theory states:

$\lambda \ll 1$: gauge-theory operators built out of free fields,
canonical dim. Δ_0 determines operators that can mix

$\lambda \gg 1$: in near-flat-space expansion string states built out of
free oscillators, level N determines states that can mix

(i) relate states with same global charges

(ii) assume direct interpolation (no “level crossing”) for states with
same quantum numbers as λ changes from small to large values

Gauge-string duality:

Konishi operator dual to

“lightest” among massive $AdS_5 \times S^5$ string states

large $\sqrt{\lambda} = \frac{R^2}{\alpha'}$:

“short” strings probe **near-flat** limit of $AdS_5 \times S^5$

members of supermultiplet:

strings with spins/oscillators in different $AdS_5 \times S^5$ directions

Flat space case:

$$m^2 = \frac{2N}{\alpha'}$$

$N = 0$: massless IIB supergravity (BPS) level

l.c. vacuum $|0\rangle$: $(8 + 8)^2 = 256$ states

$N = 2$: first massive level (many states, highly degenerate)

$$[(a_{-1}^i + S_{-1}^a)|0\rangle]^2 = [(8 + 8) \times (8 + 8)]^2$$

in $SO(9)$ reps:

$$([2, 0, 0, 0] + [0, 0, 1, 0] + [1, 0, 0, 1])^2 = (44 + 84 + 128)^2$$

$$\text{e.g. } 44 \times 44 = 1 + 36 + 44 + 450 + 495 + 910$$

$$84 \times 84 = 1 + 36 + 44 + 84 + 126 + 495 + 594 + 924 + 1980 + 2772$$

switching on $AdS_5 \times S^5$ background:

states with “lightest mass” at [first excited string level](#)

should correspond to Konishi multiplet

String spectrum in $AdS_5 \times S^5$:

long multiplets of $PSU(2, 2|4)$

highest weight states:

$$[J_2 - J_3, J_1 - J_2, J_2 + J_3]_{(s_L, s_R)}, \quad s_{L,R} = \frac{1}{2}(S_1 \pm S_2)$$

Flat-space string spectrum can be re-organized

in multiplets of $SO(2, 4) \times SO(6) \subset PSU(2, 2|4)$

[Bianchi, Morales, Samtleben 03; Beisert et al 03]

$SO(4) \times SO(5) \subset SO(9)$ rep.

lifted to $SO(4) \times SO(6)$ rep. of $SO(2, 4) \times SO(6)$

Konishi long multiplet

$$\mathcal{K} = (1 + Q + Q \wedge Q + \dots)[0, 0, 0]_{(0,0)}$$

determines the “floor” of 1-st excited string level

$$\sum_{J=0}^{\infty} [0, J, 0]_{(0,0)} \times \mathcal{K}$$

Spins: S_1, S_2 in AdS_5 ; (J_1, J_2) in S^5

orbital momentum $J = J_3$ in S^5

Examples:

- folded string with spin S_1 and momentum J :

$$S_1 = J = 2 \rightarrow [0, 2, 0]_{(1,1)}, \quad \Delta_0 = 4$$

- folded string with spin J_1 and momentum J :

$$J_1 = J = 2 \rightarrow [2, 0, 2]_{(0,0)}, \quad \Delta_0 = 4$$

- circular string with spins $J_1 = J_2$ and momentum J :

$$J_1 = J_2 = 1, J = 2 \rightarrow [0, 1, 2]_{(0,0)}, \quad \Delta_0 = 6$$

- circular string with spins $S_1 = S_2$ and momentum J :

$$S_1 = S_2 = 1, J = 2 \rightarrow [0, 1, 2]_{(0,0)}, \quad \Delta_0 = 6$$

- circular string with spins $S_1 = J_1$ and momentum J :

$$S_1 = J_1 = 1, J = 2 \rightarrow [1, 1, 1]_{(\frac{1}{2}, \frac{1}{2})}, \quad \Delta_0 = 6$$

Direct approaches to computation of quantum string energies:

(i) vertex operator approach:

use $AdS_5 \times S^5$ string sigma model perturbation theory to find leading terms in 2d anomalous dimension of corresponding vertex operators and impose marginality condition

[Polyakov 01; AT 03]

(ii) “light-cone” gauge approach:

start with AdS light-cone gauge $AdS_5 \times S^5$ string action and compute corrections to energy of corresponding flat-space oscillator string state [Metsaev, Thorn, AT 00]

both approaches yet to be developed in detail;

here will be guided by vertex operator approach

but use indirect “semiclassical” approach:

“short string” limit of semiclassical expansion

[Tirziu, AT 08; Roiban, AT 09, 11]

Target space perspective:

string in flat space: $-p^2 = m^2 = \frac{2N}{\alpha'}$

e.g. leading Regge trajectory $(\partial x \bar{\partial} x)^{S/2} e^{ipx}$, $N = S$

spectrum in (weakly) curved background:

solve marginality (1,1) conditions on vertex operators

2d anomalous dimension operator $\hat{\gamma}$ on scalar $T(x)$

differential operator in target space

from β -function for the corresponding perturbation

$$I = \frac{1}{4\pi\alpha'} \int d^2z \left[G_{mn}(x) \partial x^m \bar{\partial} x^n + T(x) \right]$$
$$\beta(T) = -2T - \alpha' \hat{\gamma} T + O(T^2)$$
$$\hat{\gamma} = \Omega^{mn} D_m D_n + \dots + \Omega^{m\dots k} D_m \dots D_k + \dots$$
$$\Omega^{mn} = \frac{1}{2} G^{mn} + O(\alpha'^3), \quad \Omega^{\dots} \sim \alpha'^n R^{\dots}$$

Solving $-\hat{\gamma} T + m^2 T = 0$ ($m^2 = -\frac{4}{\alpha'}$)

same as diagonalizing $\hat{\gamma}$

Massive string states in curved background:

$$\int d^D x \sqrt{g} \left[\Phi \dots (-D^2 + m^2 + X) \Phi \dots + \dots \right]$$
$$m^2 = \frac{2N}{\alpha'} , \quad X = R_{\dots} + O(\alpha')$$

case of $AdS_5 \times S^5$ background

$$R_{mn} - \frac{1}{96} (F_5 F_5)_{mn} = 0, \quad R = 0, \quad F_5^2 = 0$$

Find leading-order term in X ...

leading α' correction to **scalar** string state mass is 0 (!?)

$$[-D^2 + m^2 + O(\frac{1}{\sqrt{\lambda}})]\Phi = 0$$
$$\Delta = 2 + \sqrt{2N + 4 + O(\frac{1}{\sqrt{\lambda}})}$$
$$\Delta_{N=2} = 2 + 2\sqrt[4]{\lambda} \left[1 + \frac{1}{2\sqrt{\lambda}} + O(\frac{1}{(\sqrt{\lambda})^2}) \right]$$

Too naive: $SO(6)$ scalar, not 10d scalar, mixing,...

What is found for **non-singlet** (susy descendant) Konishi states?

Vertex operator approach

calculate 2d anomalous dimensions from “first principles” –
superstring theory in $AdS_5 \times S^5$:

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \left[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \right]$$

$$-Y_0^2 - Y_5^2 + Y_1^2 + \dots + Y_4^2 = -1, \quad X_1^2 + \dots + X_6^2 = 1$$

construct marginal (1,1) operators in terms of Y_p and X_k

e.g. vertex operator for dilaton (in NSR framework)

$$V_J = (Y_+)^{-\Delta} (X_x)^J \left[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \right]$$

$$Y_+ \equiv Y_0 + iY_5 = z + z^{-1} x_m x_m \sim e^{it}$$

$$X_x \equiv X_1 + iX_2 \sim e^{i\varphi}$$

$$2 = 2 + \frac{1}{2\sqrt{\lambda}} [\Delta(\Delta - 4) - J(J + 4)] + O\left(\frac{1}{(\sqrt{\lambda})^2}\right)$$

i.e. $\Delta = 4 + J$ (BPS)

Vertex operators = eigenstates of 2d anomalous dimension matrix

particular linear combinations like

$$V = f_{k_1 \dots k_\ell m_1 \dots m_{2s}} X_{k_1} \dots X_{k_\ell} \partial X_{m_1} \bar{\partial} X_{m_2} \dots \partial X_{m_{2s-1}} \bar{\partial} X_{m_{2s}}$$

their renormalization studied in $O(n)$ sigma model [Wegner 90]

simplest case: $f_{k_1 \dots k_\ell} X_{k_1} \dots X_{k_\ell}$ with traceless $f_{k_1 \dots k_\ell}$

highest-weight rep $V_J = (X_x)^J$, $\hat{\gamma} = 2 - \frac{1}{2\sqrt{\lambda}} J(J+4) + \dots$

Higher massive states:

Flat space: bosonic string state on leading Regge trajectory

$$V_S = e^{-iEt} (\partial_x \bar{\partial}_x)^{S/2}, \quad \mathbf{x} = x_1 + ix_2, \quad \alpha' E^2 = 2(S-2)$$

$AdS_5 \times S^5$: candidates for operators on leading Regge trajectory:

$$V_J = (Y_+)^{-\Delta} (\partial X_x \bar{\partial} X_x)^{J/2}, \quad X_x \equiv X_1 + iX_2$$

$$V_S = (Y_+)^{-\Delta} (\partial Y_u \bar{\partial} Y_u)^{S/2}, \quad Y_u \equiv Y_1 + iY_2$$

+ fermionic terms

+ $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms from diagonalization of anom. dim. op.

– mixing with operators with same charges and dimension.

In general $(\partial X_x \bar{\partial} X_x)^{J/2}$ mixes with singlets

$$(X_x)^{2p+2q} (\partial X_x)^{\frac{1}{2}J'-2p} (\bar{\partial} X_x)^{\frac{1}{2}J'-2q} (\partial X_m \partial X_m)^p (\bar{\partial} X_k \partial X_k)^q$$

Example of higher-level scalar/singlet operator:

$$Y_+^{-\Delta} \left[(\partial X_k \bar{\partial} X_k)^r + \dots \right], \quad N = 2(r - 1)$$

Marginality condition:

[cf. Kravtsov, Lerner, Yudson 89; Castilla, Chakravarty 96]

$$0 = 2(r - 1) - \frac{1}{2\sqrt{\lambda}} \left[\Delta(\Delta - 4) + 2r(r - 1) \right] \\ - \frac{1}{(\sqrt{\lambda})^2} \left[\frac{2}{3}r(r - 1)(r - \frac{7}{2}) + 4r \right] + \dots$$

$r = 1$: ground level– fermions should make $r = 1$ zero of $\hat{\gamma}$

$r = 2$: excited level – analog of singlet Konishi state $\Delta_0 = 2$

$$\Delta(\Delta - 4) = 4\sqrt{\lambda} - 4 + O\left(\frac{1}{\sqrt{\lambda}}\right),$$

$$\Delta - \Delta_0 = 2\sqrt[4]{\lambda} \left[1 + 0 \times \frac{1}{\sqrt{\lambda}} + O\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right]$$

fermionic contributions change subleading coefficients

Bosonic operators with momentum J and spin $J_1 \equiv J'$ in S^5 :

$$V_{J,J'} = Y_+^{-\Delta} \sum_{k,m=0}^{J'/2} c_{km} M_{km}$$

$$M_{km} \equiv X_y^{J-k-m} X_x^{k+m} (\partial X_y)^k (\partial X_x)^{\frac{1}{2}J'-k} (\bar{\partial} X_y)^m (\bar{\partial} X_x)^{\frac{1}{2}J'-m}$$

highest and lowest eigen-values of 1-loop anom. dim. matrix

$$\widehat{\gamma}_{\min} = 2 - J' + \frac{1}{2\sqrt{\lambda}} \left[\Delta(\Delta - 4) - J(J + 4) - \frac{1}{2}J'(J' + 10) - 2JJ' \right] + \dots$$

$$\widehat{\gamma}_{\max} = 2 - J' + \frac{1}{2\sqrt{\lambda}} \left[\Delta(\Delta - 4) - J(J + 4) - \frac{1}{2}J'(J' + 6) \right] + \dots$$

fermionic contributions change terms linear in J'

How to take fermionic contributions into account?

- (i) compute energies of semiclassical string states in $\frac{1}{\sqrt{\lambda}}$ expansion using full $AdS_5 \times S^5$ Green-Schwarz action
- (ii) compare to structure of $E = \Delta$ expected from marginality condition
- (iii) determine unknown coefficients in E expanded in $\frac{1}{\sqrt{\lambda}}$

General structure of dimension/energy $\Delta = E$

marginality condition – condition on quantum numbers Q_i

$$Q = (E(\lambda), S_1, S_2; J_1, J_2, J_3; \dots); \quad N = \sum_i a_i Q_i = \text{level}$$

$$0 = 2N + \frac{1}{\sqrt{\lambda}} \left(\sum_{i,j} c_{ij} Q_i Q_j + \sum_i c_i Q_i \right) \\ + \frac{1}{(\sqrt{\lambda})^2} \left(\sum_{i,j,k} c_{ijk} Q_i Q_j Q_k + \sum_{i,j} c'_{ij} Q_i Q_j + \sum_i c'_i Q_i \right) + \dots$$

- highest power of Q at n -loop order $\frac{1}{(\sqrt{\lambda})^n} Q^{n+2}$
- dependence on “momenta” E and $J = J_3$ is special
e.g., they do not enter N
- terms $\frac{1}{(\sqrt{\lambda})^n} E^m$ are scheme-dependent (cf. $(\alpha' D^2)^n$ terms in $\widehat{\gamma}$)
choose susy-preserving scheme where $E = J$ is the only solution
in BPS case – only E^2 terms are left (ignoring constant shifts)
- terms $\frac{1}{(\sqrt{\lambda})^n} E^2 Q^k$ can be traded for E -independent terms
in perturbative solution for E

States on “leading Regge trajectory”:

(max spin for given E in flat limit)

$Q = (E, J; N)$, $N =$ spin component

marginality condition:

$$0 = 2N + \frac{1}{\sqrt{\lambda}} \left(-E^2 + J^2 + n_{02}N^2 + n_{11}N \right) \\ + \frac{1}{(\sqrt{\lambda})^2} \left(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N \right) + O\left(\frac{1}{(\sqrt{\lambda})^3}\right)$$

solution for E^2 takes form [Roiban, AT 09, 11; BGMRT 12]

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N \\ + \frac{1}{\sqrt{\lambda}} \left(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N \right) \\ + \frac{1}{(\sqrt{\lambda})^2} \left(\tilde{n}_{11}J^2N + \tilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N \right) \\ + \frac{1}{(\sqrt{\lambda})^3} \left(\tilde{n}_{01}J^4N + \tilde{n}_{21}J^2N + \tilde{n}_{12}J^2N^2 + n_{05}N^5 + \dots \right) + O\left(\frac{1}{(\sqrt{\lambda})^4}\right)$$

$J^k N^m$ terms: non-decoupling of c.o.m. in curved space

Expanding in large $\sqrt{\lambda}$ for **fixed** N, J

$$E = \sqrt{2\sqrt{\lambda}N} \left[1 + \frac{A_1}{\sqrt{\lambda}} + \frac{A_2}{(\sqrt{\lambda})^2} + O\left(\frac{1}{(\sqrt{\lambda})^3}\right) \right]$$
$$A_1 = \frac{1}{4N} J^2 + \frac{1}{4} (n_{02}N + n_{11})$$
$$A_2 = -\frac{1}{2} A_1^2 + \frac{1}{4} (n_{01}J^2 + n_{03}N^2 + n_{12}N + n_{21})$$

Gives strong-coupling expansion of dimension
of corresponding gauge-theory operator

States on 1-st excited superstring level: $N = 2$

Konishi multiplet states: $N = 2, J = 2$

$$E = \sqrt[4]{\lambda} \left[2 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + O\left(\frac{1}{(\sqrt{\lambda})^3}\right) \right]$$
$$b_1 = 1 + n_{02} + \frac{1}{2}n_{11}$$
$$b_2 = -4b_1^2 + 2n_{01} + 2n_{03} + n_{12} + \frac{1}{2}n_{21}$$

To find coefficients n_{km}

use semiclassical “short string” (small spin) expansion:

- start with solitonic string carrying same charges as vertex operator representing particular quantum string state

- perform **semiclassical** expansion: $\sqrt{\lambda} \gg 1$

for **fixed** classical parameters $(\mathcal{N}, \mathcal{J})$

$$\mathcal{N} = \frac{1}{\sqrt{\lambda}} N, \quad \mathcal{J} = \frac{1}{\sqrt{\lambda}} J$$

- expand E in **small** values of \mathcal{N}, \mathcal{J}

- re-interpret the resulting E in terms of N, J

and read off n_{km}

Key point: limit $\mathcal{N} = \frac{N}{\sqrt{\lambda}} \rightarrow 0, \quad \mathcal{J} = \frac{J}{\sqrt{\lambda}} \rightarrow 0$

corresponds to $\sqrt{\lambda} \gg 1$ for fixed values of quantum charges N, J

Rewrite E^2 in terms of \mathcal{N} , \mathcal{J} :

$$\begin{aligned} \left(\frac{E}{\sqrt{\lambda}}\right)^2 &= (2\mathcal{N} + \mathcal{J}^2 + n_{01}\mathcal{J}^2\mathcal{N} + n_{02}\mathcal{N}^2 + n_{03}\mathcal{N}^3 + \tilde{n}_{01}\mathcal{J}^4\mathcal{N} + \dots) \\ &+ \frac{1}{\sqrt{\lambda}}(n_{11}\mathcal{N} + \tilde{n}_{11}\mathcal{J}^2\mathcal{N} + n_{12}\mathcal{N}^2 + \tilde{n}_{12}\mathcal{J}^2\mathcal{N}^2 + n_{13}\mathcal{N}^3 + \dots) \\ &+ \frac{1}{(\sqrt{\lambda})^2}(n_{21}\mathcal{N} + \tilde{n}_{21}\mathcal{J}^2\mathcal{N} + n_{22}\mathcal{N}^2 + \dots) + O\left(\frac{1}{(\sqrt{\lambda})^3}\right), \end{aligned}$$

- can now interpret n_{km} as semiclassical k -loop contribution to \mathcal{N}^m term in E : 1-st line is classical energy, 2nd is 1-loop, etc.
- semiclassical loop expansion ($\sqrt{\lambda} \gg 1$ for fixed \mathcal{N} , \mathcal{J}) different from loop expansion in 2d anomalous dimension– ($\sqrt{\lambda} \gg 1$ for fixed N , J) but they contain same coefficients
- each loop term in first expansion is polynomial in charges but in semiclassical expansion each term may contain infinite series in small \mathcal{J} , \mathcal{N} expansion

Semicl. expansion of E^2 organized as expansion in small \mathcal{N} or N

$$E^2 = J^2 + h_1(\lambda, J) N + h_2(\lambda, J) N^2 + h_3(\lambda, J) N^3 + \dots$$

for fixed J and large λ

$$h_1 = 2\sqrt{\lambda} + n_{11} + \frac{n_{21}}{\sqrt{\lambda}} + \frac{n_{31}}{(\sqrt{\lambda})^2} + \dots + J^2 \left(\frac{n_{01}}{\sqrt{\lambda}} + \frac{\tilde{n}_{11}}{(\sqrt{\lambda})^2} + \dots \right) + \dots$$

$$h_2 = n_{02} + \frac{n_{12}}{\sqrt{\lambda}} + \dots, \quad h_3 = \frac{n_{03}}{\sqrt{\lambda}} + \dots$$

exact “slope” h_1 for $\mathfrak{sl}(2)$ sector op. $\text{Tr}(D^S \Phi^J)$

dual to AdS_5 folded spinning string ($N = S$)

from Bethe Ansatz [Basso 11; Gromov 12, Basso 12]

$$\begin{aligned} h_1(\lambda, J) &= 2\sqrt{\lambda} \frac{d}{d\sqrt{\lambda}} \ln I_J(\sqrt{\lambda}) \\ &= 2\sqrt{\lambda} \sqrt{1 + \mathcal{J}^2} - \frac{1}{1 + \mathcal{J}^2} - \frac{\frac{1}{4} - \mathcal{J}^2}{\sqrt{\lambda}(1 + \mathcal{J}^2)^{5/2}} + \dots \\ &= 2\sqrt{\lambda + J^2} - \frac{\lambda}{\lambda + J^2} - \frac{\lambda(\frac{1}{4}\lambda - J^2)}{(\lambda + J^2)^{5/2}} + \dots \end{aligned}$$

h_1 very special: does not depend on wrapping corrections
determined by BDS form of ABA – does not depend on phase
[Can one find h_1 by direct summation of 4d or 2d
Feynmann diagrams or via localization ?]

h_1 in large J expansion:

weak coupling (gauge theory): $\lambda \ll 1, J \gg 1$

$$h_1 = 2J + \sum_{n=1}^{\infty} \frac{c_n(\lambda)}{J^n}$$

strong coupling (string theory): $\sqrt{\lambda} \gg 1, \mathcal{J} = \frac{J}{\sqrt{\lambda}} \gg 1$

$$h_1 = 2\sqrt{\lambda}\mathcal{J} + \sum_{n=1}^{\infty} \frac{a_n(\sqrt{\lambda})}{\mathcal{J}^n} = 2J + \sum_{n=1}^{\infty} \frac{\tilde{c}_n(\lambda)}{J^n}$$

Remarkably, $c_n(\lambda) = \tilde{c}_n(\lambda)$, no resummation required:
same coefficients “seen” in opposite expansions

An extension of known “non-renormalization” relations
 [Beisert,Minahan,Staudacher,Zarembo; Frolov,AT03;
 Beisert,AT 05; Frolov,Tirziu,AT 06; ...]

$$E = J + S + \sum_{n=1}^{\infty} \frac{q_n}{J^n}, \quad q_n = q_n\left(\frac{S}{J}, \lambda\right), \quad \frac{S}{J} = \frac{\mathcal{S}}{\mathcal{J}}$$

gauge theory: $\lambda \ll 1, J \gg 1$

$$q_1 = \lambda a_{11}, \quad q_2 = \lambda a_{12}, \quad q_3 = \lambda a_{13} + \lambda^2 a_{21}, \quad q_4 = \lambda a_{14} + \lambda^2 a_{22},$$

$$q_5 = \lambda a_{15} + \lambda^2 a_{23} + \lambda^3 a_{31} + \dots, \quad \dots$$

string theory: $\sqrt{\lambda} \gg 1, \mathcal{J} = \frac{J}{\sqrt{\lambda}} \gg 1$

$$q_1 = \lambda b_{00}, \quad q_2 = \lambda b_{10}, \quad q_3 = \lambda b_{20} + \lambda^2 b_{02}, \quad q_4 = \lambda b_{30} + \lambda^2 b_{12},$$

$$q_5 = \lambda^3 b_{04} + \lambda^{5/2} b_{13} + \lambda^2 b_{22} + \lambda^{3/2} b_{31} + \lambda b_{40} + \dots, \quad \dots$$

q_1, q_2, q_3, q_4 turn out to be exact: no interpolating functions in λ
 gives 6 identities between functions of $\frac{S}{J} = \frac{\mathcal{S}}{\mathcal{J}}$

(6 infinite families of coeffs match)

$$a_{11} = b_{00}, \quad a_{12} = b_{10}, \quad a_{13} = b_{20}, \quad a_{14} = b_{30}, \\ a_{21} = b_{02}, \quad a_{22} = b_{12}$$

q_5, q_6, \dots sensitive to phase in ABA: resummation required

But **linear** terms in $\frac{S}{J}$ in q_5, q_6, \dots are again special:

contribute to slope function h_1

$$E = J + \frac{1}{2}h_1(J, \lambda)\frac{S}{J} + \dots$$

Slope function in su(2) sector

[Beccaria, AT, to appear; Gromov 12 (derivation from ABA)]

Relation between folded spinning string in AdS_5 ($\mathcal{E}, \mathcal{S}; \mathcal{J}$)

and in S^5 : ($\mathcal{E}; \mathcal{J}', \mathcal{J}$) (spin $J' = J_1$)

analytic continuation [Beisert, Frolov, Staudacher, AT 03]

$$\mathcal{E} \rightarrow -\mathcal{J}, \quad \mathcal{J} \rightarrow -\mathcal{E}, \quad \mathcal{J}' \rightarrow \mathcal{S}, \quad \sqrt{\lambda} \rightarrow -\sqrt{\lambda}$$

thus $\mathfrak{su}(2)$ slope function

$$E^2 = J^2 + \tilde{h}_1(J, \lambda)J' + \dots$$

should be related to $\mathfrak{sl}(2)$ one as $\tilde{h}_1(\mathcal{J}, \sqrt{\lambda}) = -h_1(\mathcal{J}, -\sqrt{\lambda})$

$$\tilde{h}_1(J, \lambda) = -h_1(-J, -\sqrt{\lambda})$$

if $h_1(J, \lambda) = 2\sqrt{\lambda} \frac{d}{d\sqrt{\lambda}} \ln I_J(\sqrt{\lambda})$ then $\tilde{h}_1(J, \lambda) = \dots ?$

$Y_J(x) = \frac{d}{dx} \ln I_J, x \equiv \sqrt{\lambda}$, obeys

$$Y_J' = 1 + \frac{J^2}{x^2} - \frac{Y_J}{x} - Y_J^2, \quad Y_J(+\infty) = 1$$

take $x \rightarrow -x$: find solution $Z_J(x)$ to

$$-Z_J' = 1 + \frac{J^2}{x^2} + \frac{Z_J}{x} - Z_J^2, \quad Z_J(+\infty) = 1$$

boundary condition is again imposed at positive $x \rightarrow +\infty$

unique solution in terms of first Hankel function

$$Z_J(x) = -\frac{d}{dx} \ln H_J^{(1)}(i x)$$

gives correct strong-coupling expansion but

in contrast to $sl(2)$ case at weak coupling $su(2)$ slope is

ill-defined at finite J : analytic continuation to small J' required

– reflected in spurious $\ln \lambda$ terms in small λ at fixed J expansion

[wrapping corrections do not completely decouple here ?!]

To find E for quantum states one needs to know also coefficients in h_2, h_3, \dots (non-trivial, depend on wrapping corrections)

Strategy: consider examples of “small” semiclassical string states corresponding to quantum string states with angular momentum J and few oscillator (spin-carrying) modes excited

- start with classical string solutions in flat space representing states on leading Regge trajectory
- find the corresponding solutions in $AdS_5 \times S^5$
- find 1-loop correction to their energy E
- expand E in $\mathcal{N} = \frac{N}{\sqrt{\lambda}} \rightarrow 0$ – interpolate result to finite N
- find the coefficients n_{km}
- check universality of E for $N = 2$ (implied by susy)

Two basic classes of examples ($N = \text{spin}$, $J = \text{orbital momentum}$):

- **circular string with 2 spins** in two orthogonal planes
- **folded string with spin** in a plane

Rigid circular string rotating in two planes of R^4

$$t = \kappa\tau, \quad \mathbf{x}_1 \equiv x_1 + ix_2 = a e^{i(\tau+\sigma)}, \quad \mathbf{x}_2 \equiv x_3 + ix_4 = a e^{i(\tau-\sigma)}$$
$$E_{\text{flat}} = \frac{\kappa}{\alpha'} = \sqrt{\frac{2}{\alpha'}} N, \quad N = J_1 + J_2, \quad J_1 = J_2 = \frac{a^2}{\alpha'}$$

semiclassical counterpart of quantum string state created by

$$e^{-iEt} \left[(\partial_{\mathbf{x}_1} \bar{\partial}_{\mathbf{x}_1})^{\frac{J_1}{2}} (\partial_{\mathbf{x}_2} \bar{\partial}_{\mathbf{x}_2})^{\frac{J_2}{2}} + \dots \right]$$

Folded string rotating in a plane

$$t = \kappa\tau, \quad \mathbf{x}_1 \equiv x_1 + ix_2 = a \sin \sigma e^{i\tau}$$
$$E_{\text{flat}} = \sqrt{\frac{2}{\alpha'}} N, \quad N = S = \frac{a^2}{2\alpha'},$$

semiclassical counterpart of quantum string state

$$e^{-iEt} \left[(\partial_{\mathbf{x}_x} \bar{\partial}_{\mathbf{x}_x})^{S/2} + \dots \right]$$

3 ways to embed circular solutions into $AdS_5 \times S^5$:

(i) the two 2-planes in S^5 : $J_1 = J_2$ “small string”

(ii) the two 2-planes in AdS_5 : $S_1 = S_2$ “small string”

(iii) one plane in AdS_5 and the other in S^5 : $S = J'$ “small string”

2 choices – AdS_5 or S^5 – for folded string

for $N = 2$ all 5 cases represent states on 1st string level;

for $N = J = 2$ they are particular members of Konishi multiplet

can be used to check universality of λ -dependent part of $\Delta = E$

for different states in supermultiplet

Spins: S_1, S_2 in AdS_5 ; (J_1, J_2) in S^5

orbital momentum $J = J_3$ in S^5

Examples:

- folded string with spin S_1 and momentum J :

$$S_1 = J = 2 \rightarrow [0, 2, 0]_{(1,1)}, \quad \Delta_0 = 4$$

- folded string with spin J_1 and momentum J :

$$J_1 = J = 2 \rightarrow [2, 0, 2]_{(0,0)}, \quad \Delta_0 = 4$$

- circular string with spins $J_1 = J_2$ and momentum J :

$$J_1 = J_2 = 1, J = 2 \rightarrow [0, 1, 2]_{(0,0)}, \quad \Delta_0 = 6$$

- circular string with spins $S_1 = S_2$ and momentum J :

$$S_1 = S_2 = 1, J = 2 \rightarrow [0, 1, 2]_{(0,0)}, \quad \Delta_0 = 6$$

- circular string with spins $S_1 = J_1$ and momentum J :

$$S_1 = J_1 = 1, J = 2 \rightarrow [1, 1, 1]_{(\frac{1}{2}, \frac{1}{2})}, \quad \Delta_0 = 6$$

Results: for several states on leading Regge trajectory

$$\begin{aligned}
E^2 &= 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N \\
&\quad + \frac{1}{\sqrt{\lambda}}(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N) \\
&\quad + \frac{1}{(\sqrt{\lambda})^2}(\tilde{n}_{11}J^2N + \tilde{n}_{02}J^2N^2 + n_{04}N^4 + n_{13}N^3 + n_{22}N^2 + n_{31}N) \\
&\quad + \frac{1}{(\sqrt{\lambda})^3}(\tilde{n}_{01}J^4N + \tilde{n}_{21}J^2N + \tilde{n}_{12}J^2N^2 + n_{05}N^5 + \dots) + O\left(\frac{1}{(\sqrt{\lambda})^4}\right)
\end{aligned}$$

- $n_{01} = 1$, $\tilde{n}_{01} = -\frac{1}{4}$, ... from near-BMN expansion ($J \ll \sqrt{\lambda}$)

$$E^2 = J^2 + 2N\sqrt{\lambda + J^2} + \dots = J^2 + N(2\sqrt{\lambda} + \frac{J^2}{\sqrt{\lambda}} + \dots)$$

- “tree-level” coeffs $n_{02}, n_{03}, n_{04}, \dots$ are all rational
- leading 1-loop n_{11} is rational [Roiban, AT 09; Gromov et al 11]
- $\tilde{n}_{11} = -n_{11}$, i.e. in general [BGMRT 12]

$$h_1 = 2\sqrt{\lambda}\sqrt{1 + \mathcal{J}^2} + \frac{n_{11}}{1 + \mathcal{J}^2} + \frac{1}{\sqrt{\lambda}}[n_{21} + \tilde{n}_{21}\mathcal{J}^2 + O(\mathcal{J}^4)] + O\left(\frac{1}{(\sqrt{\lambda})^2}\right)$$

$$h_2 = \frac{n_{02} + \mathcal{J}^2}{1 + \mathcal{J}^2} + \frac{1}{\sqrt{\lambda}}[n_{12} + \tilde{n}_{12}\mathcal{J}^2 + O(\mathcal{J}^4)] + O\left(\frac{1}{(\sqrt{\lambda})^2}\right)$$

- $n_{12} = n'_{12} - 3\zeta_3$, $n'_{12} = -\frac{3}{8} - 2n_{03}$ is rational

[Tirziu, AT 08; Roiban, AT 09; Gromov-Valatka 11]

ζ_3 term is **universal** for states on leading Regge trajectory

- $\tilde{n}_{12} = \tilde{n}'_{12} + 3\zeta_3 + \frac{15}{4}\zeta_5$, \tilde{n}'_{12} rational

- n_{1k} contains universal ζ_{2k-1} (universal UV $n \gg 1$ asymptotics)

e.g. $n_{13} = \tilde{n}'_{12} + \tilde{n}''_{1k}\zeta_3 + \frac{15}{4}\zeta_5$

- leading 2-loop coefficient n_{21} is **universal**: $n_{21} = -\frac{1}{4}$

for folded string state [Basso]; evidence from universality [BGMRT]

of the Konishi state energy ($J = N = 2$)

$$E_{N=J=2} = \sqrt[4]{\lambda} \left[2 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + \frac{b_3}{(\sqrt{\lambda})^3} + \dots \right]$$

$$b_1 = 1 + n_{02} + \frac{1}{2}n_{11} = 2$$

$$b_2 = -\frac{1}{4}b_1^2 + 2n_{01} + 2n_{03} + n_{12} + \frac{1}{2}n_{21} = \frac{1}{2} - 3\zeta_3$$

$$b_3 = a_1 + a_2\zeta_3 + \frac{15}{2}\zeta_5, \quad \dots$$

b_1, b_2 : match TBA predictions interpolated to $\lambda \gg 1$

- need 2-loop string sigma model computation

to confirm universality of n_{21} , fix $n_{22} \rightarrow$ determine b_3

Some details:

- Circular rotating string in S^5 with $J_1 = J_2 \equiv J'$:

flat space $R_t \times R^4$: circular string solution

$$x_1 + ix_2 = a e^{i(\tau+\sigma)}, \quad x_3 + ix_4 = a e^{i(\tau-\sigma)}$$

$$E = \sqrt{\frac{4}{\alpha'} J'}, \quad J' = \frac{a^2}{\alpha'}$$

directly embedded into $R_t \times S^5$ in $AdS_5 \times S^5$ [Frolov, AT 03]:

string on **small** sphere inside S^5 : $X_1^2 + \dots + X_6^2 = 1$

$$\begin{aligned} X_1 + iX_2 &= a e^{i(\tau+\sigma)}, & X_3 + iX_4 &= a e^{i(\tau-\sigma)}, \\ X_5 + iX_6 &= \sqrt{1 - 2a^2}, & t &= \kappa\tau \\ \mathcal{J}' = \mathcal{J}_1 = \mathcal{J}_2 &= a^2, & \mathcal{E}_0^2 = \kappa^2 &= 4\mathcal{J}' \end{aligned}$$

E_0 is just as in flat space

$$E_0 = \sqrt{\lambda} \mathcal{E} = \sqrt{4\sqrt{\lambda} J'} , \quad J' = \sqrt{\lambda} \mathcal{J}'$$

1-loop correction to energy: closed string ($R \times S^1$)

sum of bosonic and fermionic fluctuation frequencies ($n = 0, 1, 2, \dots$)

Bosons (2 massless + massive):

$$AdS_5 : \quad 4 \times \quad \omega_n^2 = n^2 + 4\mathcal{J}'$$

$$S^5 : \quad 2 \times \quad \omega_{n\pm}^2 = n^2 + 4(1 - \mathcal{J}') \pm 2\sqrt{4(1 - \mathcal{J}')n^2 + 4\mathcal{J}'^2}$$

Fermions:

$$4 \times \quad \omega_{n\pm}^{2f} = n^2 + 1 + \mathcal{J}' \pm \sqrt{4(1 - \mathcal{J}')n^2 + 4\mathcal{J}'^2}$$

$$E_1 = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \left[4\omega_n + 2(\omega_{n+} + \omega_{n-}) - 4(\omega_{n+}^f + \omega_{n-}^f) \right]$$

expand in small \mathcal{J}' and do sums (ζ_k come from \sum_n)

$$E_1 = \frac{1}{\sqrt{\mathcal{J}'}} \left[\mathcal{J}' - (3 + \zeta_3)\mathcal{J}'^2 - \frac{1}{4}(5 + 6\zeta_3 + 30\zeta_5)\mathcal{J}'^3 + \dots \right]$$

$$E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda}J'} \left[1 + \frac{1}{2\sqrt{\lambda}} - \frac{3}{4}(1 + 2\zeta_3) \frac{J'}{(\sqrt{\lambda})^2} + \dots \right]$$

To get a state on the first excited string level ($N = 2J'$)

should choose $J' = 1$, i.e. $J_1 = J_2 = 1$

for minimal non-trivial value of $J = J_3 = 2$

there is unique corresponding state in Konishi multiplet table:

$[0, 1, 2]_{(0,0)}$ at level $\Delta_0 = 6$ and thus

$$b_1 = 2 \left(\frac{J^2}{8J'} + \frac{1}{2} \right)_{J=2, J'=1} = 2$$

- **Small circular spinning string with $S_1 = S_2$**

rigid circular string with two equal spins in AdS^5

and orbital momentum $J = J_3$ in S^5

$$\begin{aligned} Y_0 + iY_5 &= \sqrt{1 + 2r^2} e^{i\kappa t}, & Y_1 + iY_2 &= r e^{i(w\tau + \sigma)}, & Y_3 + iY_4 &= r e^{i(w\tau - \sigma)} \\ X_5 + iX_6 &= e^{i\nu\tau}, & w^2 &= \kappa^2 + 1, & \kappa^2(1 + 2r^2) &= 2r^2(1 + w^2) + \nu^2 \end{aligned}$$

$$\mathcal{E}_0 = (1 + 2r^2)\kappa = \kappa + \frac{2\kappa\mathcal{S}}{\sqrt{1 + \kappa^2}}, \quad \mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2 = r^2 w, \quad \mathcal{J} = \nu$$

“short” string expansion of the classical energy

$$E_0 = \sqrt{\lambda}\mathcal{E}_0, \quad \mathcal{E}_0 = 2\sqrt{\mathcal{S}} \left(1 + \mathcal{S} + \frac{\mathcal{J}^2}{8\mathcal{S}} + \dots \right)$$

including 1-loop correction:

$$E_0 + E_1 = 2\sqrt{\sqrt{\lambda}\mathcal{S}} \left[1 + \frac{1}{\sqrt{\lambda}} \left(\mathcal{S} + \frac{\mathcal{J}^2}{8\mathcal{S}} - \frac{1}{2} \right) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right]$$

state on the first excited level ($N = 2S$)

has two excited oscillators, i.e. should have $S = S_1 = S_2 = 1$

for $J = 2$ the dual state in representation $[0, 2, 0]_{(1,0)}$

there is just one state in Konishi table with $\Delta_0 = 6$

$$b_1 = 2 \left(S + \frac{J^2}{8S} - \frac{1}{2} \right)_{S=1, J=2} = 2$$

- Small circular spinning string with $S = J_1$ and $J = J_2 \neq 0$

rigid circular solution with one spin in AdS_5 and one spin in S^5
and orbital momentum J in S^5

$$\begin{aligned}
 Y_0 + iY_5 &= \sqrt{1 + r^2} e^{i\kappa t}, & Y_1 + iY_2 &= r e^{i(w\tau + \sigma)}, & w^2 &= \kappa^2 + 1, \\
 X_1 + iX_2 &= a e^{i(w'\tau - \sigma)}, & X_3 + iX_4 &= \sqrt{1 - a^2} e^{i\nu\tau}, & w'^2 &= \nu^2 + 1, \\
 \mathcal{E}_0 &= 2\sqrt{S} \left(1 + \frac{1}{2}S + \frac{J^2}{8S} + \dots \right)
 \end{aligned}$$

The leading 1-loop correction to the energy vanishes
(cancellation of AdS and sphere contributions)

$$E_0 + E_1 = 2\sqrt{\sqrt{\lambda}S} \left[1 + \frac{1}{\sqrt{\lambda}} \left(\frac{1}{2}S + \frac{J^2}{8S} \right) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right]$$

state on the first excited level: $S = J_1 = 1$

for $J = 2$ get state $[1, 1, 1]_{(\frac{1}{2}, \frac{1}{2})}$ at $\Delta_0 = 6$ level

$$b_1 = 2 \left(\frac{1}{2}S + \frac{J^2}{8S} \right)_{S=1, J=2} = 2$$

Conclusions

- progress in understanding of $AdS_5 \times S^5$ string spectrum
= spectrum of conformal $\mathcal{N} = 4$ SYM operators
- agreement with numerical results from TBA:
non-trivial check of quantum string integrability
- prediction of transcendental structure of leading coefficients:
reproduce them by an analytic solution of TBA at strong coupling?
- evidence of universality of some coefficients in strong coupling
expansion of dimensions of states on leading Regge trajectory
- need systematic study of quantum string theory in $AdS_5 \times S^5$
in near-flat-space expansion

... and still need **first-principles** solution for the
spectrum of $AdS_5 \times S^5$ superstring = spectrum of $\mathcal{N} = 4$ SYM
based on **integrability**