Refined Donaldson-Thomas theory and Nekrasov's formula

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- Integrating out the $\mathbb{R}^4$-directions results in a $U(1)$ gauge theory on the threefold X. The partition function Z_X is (a version of) the topological string partition function of X
- The aim of the talk is to discuss the precise relationship

$$Z_{\mathbb{C}^2} \sim Z_X$$

in the simplest example, and study the question whether there is anything more to this than an equality of generating series.

Gauge-theoretic moduli spaces on $\mathbb{C}^2$

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$$\mathcal{M}^{\circ}_{n,k} = \left\{ (E, \nabla) \text{ framed finite action } U(n)\text{-instanton of charge } k \text{ on } \mathbb{R}^4 \right\}.$$

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By the Kobayashi–Hitchin correspondence (Donaldson), this space can also be obtained as the moduli space

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Here $\mathbb{C}^2 \subset \mathbb{P}^2$ with complement $l_\infty \cong \mathbb{P}^1$, and the framing is an isomorphism

$$\phi \colon \mathcal{E}|_{l_\infty} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$$

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This is an ill-defined expression.

Symmetries of the gauge-theoretic moduli spaces

Note that all $\mathcal{M}_{n,k}$ carry an action of the torus $T = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^{n-1}$.

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Thus T-equivariant integrals make sense on the moduli space $\mathcal{M}_{n,k}$.

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Thus Nekrasov defines

$$Z_{\mathbb{C}^2}^{U(n)}(\Lambda, q_1, q_2, a_1, \ldots, a_{n-1}) = \sum_{k \geq 0} \Lambda^k \, \mathrm{char}_T \, H^*(\mathcal{M}_{n,k}, \mathcal{O}_{\mathcal{M}_{n,k}})$$

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Here, for a representation V of T, $\operatorname{char}_T V \in \mathbb{Z}[q_i, a_j]$ denotes its T-character.

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and that

$$H^0\left(\mathrm{Hilb}^k(\mathbb{C}^2), \mathcal{O}\right) \cong H^0\left(S^k(\mathbb{C}^2), \mathcal{O}\right)$$

where $S^k(\mathbb{C}^2)$ is the k-th symmetric power of $\mathbb{C}^2$.

Nekrasov's partition function for $U(1)$: the computation

Now we can finish the computation:

$$Z_{\mathbb{C}^2}^{U(1)}(\Lambda, q_1, q_2) \; = \; \sum_{k \geq 0} \Lambda^k \, \mathrm{char}_T \, H^* \left(\mathrm{Hilb}^k(\mathbb{C}^2), \mathcal{O} \right)$$

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Relation to the conifold

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Indeed,

$$-\log Z^{U(1)}_{\mathbb{C}^2}\left(\Lambda, q_1 = q_2 = e^{i\hbar}\right) = \sum_{g\geq 0} \hbar^{2g-2} \mathcal{F}^X_g(T)$$

where $\mathcal{F}^X_g(T)$ is the genus-g Gromov-Witten potential of X.

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The corresponding moduli spaces $\mathcal{N}_{n,l}$ are singular gauge-theoretic moduli spaces associated to X.

More about gauge-theoretic moduli spaces on X

Recall that $Z'_X(q, T)$ was a specialization of the full $U(1)$ partition function of $\mathbb{C}^2$ at $q_1 = q_2 = q$. It is natural to ask what is the geometric interpretation of the full partition function on the conifold X.

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Theorem (Sz., Nagao-Nakajima) The spaces $\mathcal{N}_{n,l}$ are global critical loci

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This uses the quiver description of the conifold and the Klebanov-Witten superpotential.

Refining the numerical gauge theoretic invariants of X

Using the critical locus interpretation, one gets a topological coefficient system $\phi_{n,l}$ on the singular moduli spaces $\mathcal{N}_{n,l}$, and a corresponding cohomology theory with mixed Hodge structure

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This weight polynomial refinement of the Euler characteristic is equivalent to the motivic refinement introduced by Kontsevich-Soibelman and studied by Behrend-Bryan-Sz., Dimofte-Gukov and others.

Interpretation of the full partition function on X

Let

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Thus we obtain a cohomological interpretation on X of the full Nekrasov partition function in this case.

Relationship between the underlying vector spaces

Recall from the computation that

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Assuming a purity conjecture, this suggests that there should be a natural isomorphism

$$\oplus_{n,l} H^*(\mathcal{N}_{n,l}, \phi_{n,l}) \cong \Lambda^*\mathbb{C}[x,y].$$

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just the Hilbert series of a triply-graded vector space, the symmetric space of the space $\mathbb{C}[x,y]$ of functions on $\mathbb{C}^2$. Then,

$$Z_{\mathbb{C}^2}^{U(1)}(\Lambda, q_1, q_2)^{-1} = \mathrm{char}_{T\times\mathbb{C}^*} \Lambda^*\mathbb{C}[x,y],$$

the corresponding exterior space!

Assuming a purity conjecture, this suggests that there should be a natural isomorphism

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Relationship between the underlying vector spaces

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I don't know how to construct this, but SL(2)-action on the left hand side... should come from a Lefschetz SL(2)-action on the cohomology!

Towards higher rank invariants of the conifold

Keep X to be the resolved conifold, but consider geometric engineering on $Y \times X$ instead, where $Y \to \mathbb{C}^2/\mathbb{Z}_n$ is the minimal resolution of the Type A_{n-1} surface singularity (Nekrasov).

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On the conifold X, this should correspond to some version of $U(n)$ Donaldson-Thomas theory. I don't know an interpretation at present!