Gravity duals of supersymmetric gauge theories on curved manifolds

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Based on work with Dario Martelli, Achilleas Passias

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In this talk I will describe gauge/gravity duality for such gauge theories with a large **N** limit.

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Vector multiplet $(\mathcal{A}_\mu, \sigma, \chi, \mathsf{D})$, $\sigma =$ adjoint scalar,

$$\mathsf{S}_{\mathrm{CS}} = \frac{\mathsf{k}}{4\pi} \int \mathrm{Tr} \left(\mathcal{A} \wedge \mathrm{d}\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} - \bar{\chi}\chi + 2\mathsf{D}\sigma \right) .$$

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Matter in chiral multiplets (ϕ, ψ, F), in representation $\mathcal{R}$ of gauge group $\mathbf{G}$, with R-charge Δ.

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One can also put on other Riemannian 3-manifolds, with appropriate background R-symmetry gauge field $\mathbf{A^{(3)}}$:

$$\text{Flat space} \quad \partial_\mu - \mathrm{i}q\mathcal{A}_\mu \quad \longrightarrow \quad \nabla_\mu - \mathrm{i}q\mathcal{A}_\mu - \mathrm{i}\Delta \mathbf{A^{(3)}}_\mu \ .$$

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Systematic approach: begin with supergravity, take an appropriate $\mathbf{m_{pl}} \to \infty$ limit to obtain a rigid SUSY theory. Gravitino SUSY equation $\Rightarrow$ Killing spinor equation [Festuccia-Seiberg]:

$$\text{Flat space} \quad \partial_\mu \epsilon \ = \ \mathbf{0} \quad \longrightarrow \quad \nabla_\mu \epsilon - \mathbf{iA^{(3)}}_\mu \epsilon \ = \ \cdots \ .$$

As an example we will consider a Berger squashed sphere

$$\mathrm{d}s_3^2 \;=\; \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 + \frac{1}{v^2}\left(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi\right)^2 \;,$$

with background R-symmetry gauge fields

$$\text{(i)}\quad A^{(3)} \;=\; \frac{v^2-1}{2v^2}\left(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi\right) \qquad \text{[Hama-Hosomichi-Lee]} \;,$$

$$\text{(ii)}\quad A^{(3)} \;=\; \frac{\sqrt{v^2-1}}{2v^2}\left(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi\right) \qquad \text{[Imamura-Yokoyama]} \;.$$

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The Killing spinor in case (i) is an **SU(2)** singlet, while case (ii) is a doublet.

A key point is that the VEV of any BPS operator *localizes*

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\langle\, \mathcal{O}_{\mathsf{BPS}} \,\rangle \quad &= \quad \int_{\text{all fields}} \mathrm{e}^{-\mathsf{S}}\, \mathcal{O}_{\mathsf{BPS}} \\
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For $\mathbf{d = 3}$, $\mathcal{N} = \mathbf{2}$ $\mathbf{U(N)}$ gauge theory, infinite-dimensional functional integral $\longrightarrow$ *finite-dimensional* integral over zero-mode of $\boldsymbol{\sigma} = \mathbf{N} \times \mathbf{N}$ Hermitian matrix.

Example: Consider the partition function $\mathbf{Z} = \langle \mathbf{1} \rangle$ for case (ii), for a $\mathbf{U(N)_k}$ Chern-Simons gauge theory with a single fundamental field, R-charge $\boldsymbol{\Delta}$:

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$$\mathbf{Z[v,N,k]} \;=\; \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \cdot \exp\left[\frac{i\pi k}{v^2}\sum_{i=1}^{N}\lambda_i^2\right] \cdot \frac{\prod_{i\neq j} s_b\left[\frac{1}{v}(\lambda_i - \lambda_j - i)\right]}{\prod_{i=1}^{N} s_b\left[\frac{1}{v}(\lambda_i - i(1-\Delta))\right]}$$

Here $\boldsymbol{\lambda_i}$, $\mathbf{i = 1, \ldots, N}$, are the real eigenvalues of $\boldsymbol{\sigma}$, $\mathbf{b} = \dfrac{1 + i\sqrt{v^2 - 1}}{v}$, and $\mathbf{s_b}$ = quantum dilogarithm function.

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So far, everything is valid for an arbitrary $\mathbf{d = 3}$, $\mathcal{N} = \mathbf{2}$ gauge theory.

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The obvious candidate is the $\mathcal{N} = \mathbf{6}$ ABJM theory: a $\mathbf{U(N)_k} \times \mathbf{U(N)_{-k}}$ Chern-Simons-matter theory, with 2 bifundamentals in each of $\mathbf{(\bar{N}, N)}$, $\mathbf{(N, \bar{N})}$.

On the round $\mathbf{S^3}$, in the fixed $\mathbf{k}$, $\mathbf{N} \to \infty$ limit, this is dual to the $\mathbf{d = 11}$ SUGRA background $\mathbf{AdS_4} \times \mathbf{S^7}/\mathbb{Z}_\mathbf{k}$ with $\mathbf{N}$ units of flux.

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$$-\log Z_{\text{field theory}} \; = \; \frac{\pi\sqrt{2}}{3} k^{1/2} N^{3/2} + o(N^{3/2}) \; = \; -\log Z_{\text{SUGRA}} \, .$$

[Drukker-Marino-Putrov]

One can modify $\mathbf{AdS_4} \times \mathbf{S^7}$ in two obvious ways:

- replace $\mathbf{S^7}$ by a more general Sasaki-Einstein manifold $\mathbf{Y_7}$,
- replace $\mathbf{AdS_4}$ by a more general Einstein manifold $\mathbf{M_4}$, conformal boundary $\mathbf{M_3} = \partial \mathbf{M_4}$.

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- putting the gauge theory on the 3-manifold $\mathbf{M_3}$.

N.B. In general, must sum over different $\mathbf{M_4}$ with fixed data on $\mathbf{M_3} = \partial \mathbf{M_4}$.

We may engineer $\mathbf{d = 3}$, $\mathcal{N} = \mathbf{2}$ gauge theories with a large $\mathbf{N}$ gravity dual by putting $\mathbf{N}$ M2-branes at a Calabi-Yau 4-fold singularity.

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Perhaps the simplest non-trivial example is the quadric singularity in $\mathbb{C}^{\mathbf{5}}$:

$$\sum_{i=1}^{5} z_i^2 = 0 .$$

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The (duality frame of the) dual gauge theory depends on a choice of M-theory circle. One choice leads to a $\mathbf{U(N)_k \times U(N)_{-k}}$ theory, with 2 bifundamentals in each of $(\bar{\mathbf{N}}, \mathbf{N})$, $(\mathbf{N}, \bar{\mathbf{N}})$, 2 adjoints, and a cubic superpotential. [Martelli-JFS]

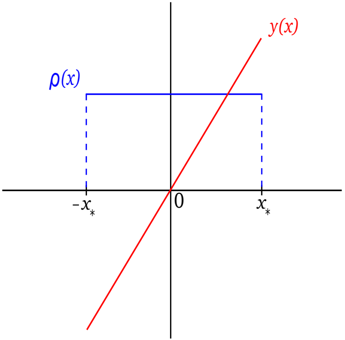
We solve the large **N** matrix integral using a saddle point method.

Leading order complex saddle point solution

$$\lambda_{1,2}(x) \;=\; N^{1/2}x \pm i y(x)\;,$$

where the eigenvalue density $\rho(x)$ is constant.

$$x_* \;=\; \frac{4\pi}{3\sqrt{k}}\,, \quad y(x_*) \;=\; \frac{2\pi}{3}\;.$$



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The gauge/gravity dual prediction is that this should equal

$$-\log Z_{\text{SUGRA}} = S_{\text{gravity}}(AdS_4) = N^{3/2}\sqrt{\frac{2\pi^6}{27\text{Vol}(Y_7)}} \ ,$$

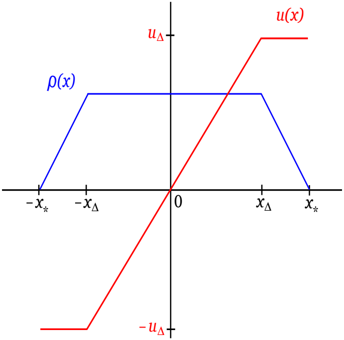
and it is straightforward to check agreement for $Y_7 = SO(5)/SO(3)$! [Martelli-JFS], but see also [Jafferis-Klebanov-Pufu-Safdi], [Cheon-Kim-Kim]. This week: [Amarati-Franco]

Straightforward to extend to other examples (provided they are *non-chiral*).

$\mathbf{U(N)_{2k} \times U(N)_{-k} \times U(N)_{-k}}$ gauge theory with 7 matter fields.

Dual to **N** M2-branes at a certain toric Calabi-Yau 4-fold singularity.

$$-\log Z = \frac{32\pi N^{3/2}}{3}(2-\Delta)(1-\Delta) \times \sqrt{\frac{\Delta}{4-3\Delta}} + o(N^{3/2}) \,,$$



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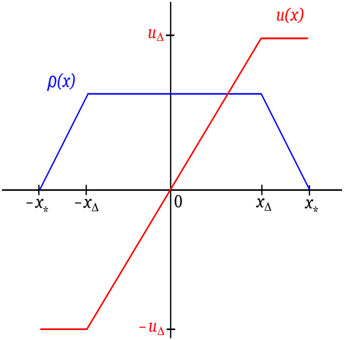

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where $\Delta = \frac{1}{18}\left[19 - \frac{37}{(431-18\sqrt{417})^{1/3}} - (431-18\sqrt{417})^{1/3}\right]$.

One can also match the VEV of the BPS Wilson loop

$$\mathbf{W} = \mathrm{Tr}\left[\mathcal{P}\exp\left(\oint \mathrm{d}\tau(\mathrm{i}\mathcal{A}_\mu \dot{\mathbf{x}}^\mu + \sigma|\dot{\mathbf{x}}|)\right)\right] ,$$

where we integrate around a Hopf fibre $\mathbf{S}^1 \subset \mathbf{S}^3$. This translates to the matrix model

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This can be reproduced from the action of a BPS membrane wrapping the M-theory circle, and a minimal surface in AdS with boundary the Hopf circle.

[Martelli-Passias-JFS] work in progress

We now turn to deforming $\mathbf{S^3} \longrightarrow$ squashed $\mathbf{S^3}$. Recall we had two choices of background R-symmetry gauge fields (i), (ii). One computes the large $\mathbf{N}$ result

$$\log \mathbf{Z}_{\text{field theory}}[\mathbf{v}] = \log \mathbf{Z}_{S^3} \times \begin{cases} \mathbf{1} & \text{case (i)} \\ \frac{\mathbf{1}}{\mathbf{v}^2} & \text{case (ii)} \end{cases} .$$

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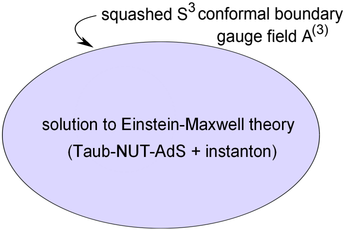
$$\log Z_{\text{field theory}}[\mathbf{v}] = \log Z_{S^3} \times \begin{cases} 1 & \text{case (i)} \\ \frac{1}{v^2} & \text{case (ii)} \end{cases} .$$

We seek the gravity dual $\mathbf{M_4}$ as a solution to $\mathbf{d = 4}$, $\mathcal{N} = \mathbf{2}$ gauged SUGRA (Einstein-Maxwell theory).

Fact: Any SUSY solution of this theory uplifts to an exact SUSY solution of $\mathbf{d = 11}$ SUGRA [Gauntlett-Varela].

This is a Dirichlet problem:



Solution: $\mathbf{M_4} = \text{Taub-NUT-AdS}$

(i) self-dual $\mathbf{U(1)}$ gauge field ,

(ii) anti-self-dual $\mathbf{U(1)}$ gauge field .

We also find $\epsilon_{\mathbf{4d}} \mid_{\partial \mathbf{M_4}} = \epsilon_{\mathbf{3d}}$.

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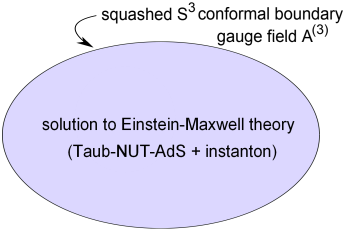

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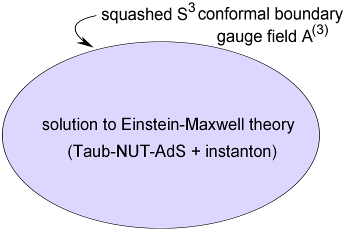

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$$-\log Z_{\text{SUGRA}} \;=\; S_{\text{Einstein-Hilbert}} + S_{\text{Gibbons-Hawking}} + S_{\text{counterterm}} \;.$$

Precisely reproduces the large **N** field theory results! In particular, in case (i) although each term depends on the squashing parameter **v**, the total expression is **v**-independent!

The solutions:

$$\mathrm{ds}^2_{\mathrm{Taub-NUT-AdS}} = \frac{(r^2 - \frac{1}{(2v)^2})}{\Omega(r)}\mathrm{d}r^2 + \left(r^2 - \frac{1}{(2v)^2}\right)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + \frac{\Omega(r)}{v^2(r^2 - \frac{1}{(2v)^2})}(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi)^2 ,$$

where

$$\Omega(r) = \left(r - \frac{1}{(2v)^2}\right)^2 \left[1 + \left(r - \frac{1}{2v}\right)\left(r + \frac{3}{2v}\right)\right] .$$

Gauge fields

$$\text{(i)} \quad A = \frac{(v^2 - 1)(r - \frac{1}{2v})}{2v^2(r + \frac{1}{2v})}(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi) ,$$

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- Topology $\mathcal{O}(-1) \to \mathbf{S^2}$, while Taub-NUT-AdS has topology $\mathbb{R}^4$.
- They exist (as smooth, real solutions) only for certain ranges of squashing $\mathbf{v}$.

We refer to these as Taub-Bolt-AdS$_\pm$ [N.B. there is an Einstein Taub-Bolt-AdS metric, which is different].

Some further interesting subtleties:

- Taub-Bolt-AdS$_\pm$ solutions are not spin manifolds, but we find automatically $\int_{S^2} \frac{F}{2\pi} = \pm\frac{1}{2}$, which means there is a well-defined spin^c structure.

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- Only uplift to regular $\mathbf{d = 11}$ SUGRA solutions when $\mathbf{Y_7}$ is a *regular* Sasaki-Einstein manifold (*i.e.* the R-symmetry is globally $\mathbf{U(1)_R}$).

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Some further interesting subtleties:

- Taub-Bolt-AdS$_\pm$ solutions are not spin manifolds, but we find automatically $\int_{S^2} \frac{F}{2\pi} = \pm\frac{1}{2}$, which means there is a well-defined spin^c structure.
- Only uplift to regular $d = 11$ SUGRA solutions when Y_7 is a *regular* Sasaki-Einstein manifold (*i.e.* the R-symmetry is globally $U(1)_R$).
- Subleading terms in the metric and gauge field are different.

Although the metrics are non-Einstein, and very complicated, we again find that the gravitational action is independent of $\mathbf{v}$:

Although the metrics are non-Einstein, and very complicated, we again find that the gravitational action is independent of $\mathbf{v}$:

$$\log Z_{\mathrm{SUGRA}}[\mathbf{v}] \;=\; \log Z_{S^3} \times \begin{cases} \frac{3}{4} & \text{Taub-Bolt-AdS}_+ \\ 1 & \text{Taub-NUT-AdS} \\ \frac{5}{4} & \text{Taub-Bolt-AdS}_- \end{cases} .$$

The Taub-Bolt-AdS$_+$ solution has smaller free energy!