Adventures with Contact Terms

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Outline

Throughout the talk we will be in $3d$.

- Chern-Simons contact terms
- Currents in $\mathcal{N} = 2$ supersymmetry
- Various SUSY Chern-Simons contact terms
- Anomaly: superconformal vs. (compact) $U(1)_R$ symmetry
- $\mathcal{N} = 2$ on a three sphere
- The sphere partition function
- F-maximization
- Conclusions

Contact Terms

- Contact terms are correlation functions at coincident points.
- Some of them are determined; e.g. the seagull term (needed for gauge invariance), the $2d$ conformal anomaly (in a CFT T_μ^μ is a redundant operator, but it must have nonzero contact terms).
- Most of them are arbitrary (not universal).
  - They reflect short distance physics.
  - They depend on the regularization scheme.
  - They are associated with local counter terms constructed out of the dynamical fields and background fields.
  - They change under coupling constant redefinitions.

An Important Exception

Consider a three-dimensional field theory with a global (compact) $U(1)$ symmetry.
The conserved current j_μ can be coupled to a classical background $U(1)$ gauge field a_μ.
A contact term in the two-point function

$$\langle j_\mu(x) j_\nu(0) \rangle = \cdots + \frac{i\kappa}{2\pi} \epsilon_{\mu\nu\rho} \partial^\rho \delta^{(3)}(x)$$

can be interpreted as due to a local counter term in the background fields

$$\frac{i\kappa}{4\pi} \int \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \ .$$

$$\frac{i\kappa}{4\pi}\int \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$$

- The contact term dogma is that κ is arbitrary. It changes when we change the regularization, and we have freedom to change it arbitrarily – add a bare counter term.
- However, since $U(1)$ is compact, and we would like the theory to make sense on arbitrary manifolds with arbitrary $U(1)$ bundles, the freedom in κ is quantized.
- More explicitly, a consistent definition of the theory forces us to add such a bare counter term with fixed value of $\kappa \bmod(1)$; only the integer part of κ is arbitrary.

The Current Two-Point Function

Current conservation restricts

$$\int e^{ip\cdot x} \langle j_\mu(x) j_\nu(0)\rangle = \tau\left(\frac{p^2}{\mu^2}\right)\frac{p_\mu p_\nu - p^2\delta_{\mu\nu}}{16|p|} + \kappa\left(\frac{p^2}{\mu^2}\right)\frac{\varepsilon_{\mu\nu\rho}p^\rho}{2\pi}$$

- Two structure functions
- In a conformal field theory τ and κ are independent of p.
- In a unitary CFT $\tau > 0$.
- The p dependence of $\kappa\left(\frac{p^2}{\mu^2}\right)$ is physical – not a contact term.
- Shifting $\kappa\left(\frac{p^2}{\mu^2}\right)$ by a constant amounts to adding a contact term.
- If the symmetry is compact, this ambiguity is quantized.

We define

$$\kappa_{UV} \equiv \lim_{p\to\infty} \kappa\left(\frac{p^2}{\mu^2}\right)$$

$$\kappa_{IR} \equiv \lim_{p\to 0} \kappa\left(\frac{p^2}{\mu^2}\right)$$

They can be changed by adding a local counter term, but $\kappa_{UV} - \kappa_{IR}$ is physical.

The same story can be repeated for the Lorentz Chern-Simons term

$$\frac{i}{4}\int \varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\omega_\mu\partial_\nu\omega_\rho + \frac{2}{3}\omega_\mu\omega_\nu\omega_\rho\right)$$

and the related “framing dependence.”

Example 1: Free Fermions

A theory with a single massive fermion has a global $U(1)$ symmetry with [Redlich]

$$\kappa_{UV} - \kappa_{IR} = \frac{1}{2}\mathrm{sign}(m) \ .$$

The IR theory has no degrees of freedom.
The effective Lagrangian is proportional to a Chern-Simons term for the background $U(1)$ gauge field.

Consistency demands $\kappa_{IR} \in \mathbf{Z}$, and therefore

$$\kappa_{UV} = \frac{1}{2} + \mathrm{integer} \ .$$

Example 2: A Topological theory

Consider a theory of a dynamical $U(1)$ gauge field A_μ and a classical $U(1)$ gauge field a_μ

$$\mathcal{L} = \frac{i}{4\pi}\left(k\,\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + 2p\,\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho + q\,\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho\right)$$

with $k, p, q \in \mathbf{Z}$.
Naively integrating out A_μ, we find the effective theory

$$\mathcal{L}_{eff} = \frac{i\kappa}{4\pi}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho$$

$$\kappa = \kappa_{UV} = \kappa_{IR} = q - \frac{p^2}{k}$$

- κ is fractional, but the theory is consistent.
- There are remaining topological IR degrees of freedom. Hence, A_μ was not integrated out properly.

Example 3: Flowing From a Free Theory to a Massive Theory

Computing in perturbation theory and taking into account only 1PI graphs:

- The subtlety in the previous example does not affect κ^{1PI}.
- κ_{UV}^{1PI} must be integer or half-integer.
- Non-renormalization theorem [Coleman, Hill]:
  κ_{IR}^{1PI} can be generated only at one loop.
  Alternatively, κ_{IR}^{1PI} must be quantized and then the theorem follows from gauge invariance.

Example 4: Interesting Renormalization Group Flow

We often have a free theory in the UV.
The IR theory is fully gapped (not even topological d.o.f).
At energies E such that $m \ll E \ll M$ the theory is approximately conformal. M and m are crossover scales.

- $\kappa_{UV} = \lim_{p\to\infty} \kappa(p^2)$ is determined modulo an integer by the number of fermions and the coupling to topological terms.
- $\kappa_{CFT} = \kappa(m^2 \ll p^2 \ll M^2)$ is an intrinsic observable of the CFT – it is well defined modulo an integer.
- $\kappa_{IR} = \lim_{p\to 0} \kappa(p^2)$ must be quantized.

Currents in $\mathcal{N} = 2$ Supersymmetry in $3d$

Supersymmetrizing the previous discussion, we distinguish between ordinary global symmetries and an R-symmetry.

A global non-R $U(1)$ symmetry can be coupled to a classical background $U(1)$ gauge superfield: $(a_\mu, \sigma, D, \lambda_\alpha)$.
The "auxiliary field" D is not constrained to satisfy its equation of motion.
The supersymmetric Chern-Simons term is

$$\frac{\kappa}{4\pi}\left(i\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho - 2\sigma D + \text{fermions}\right) \ .$$

Again, κ is well defined up to an additive constant.
For a compact symmetry group this constant is quantized.

R-Symmetry

The $U(1)_R$ current is in the same supermultiplet as the energy momentum tensor and the supersymmetry current.
The classical background fields are the metric $g_{\mu\nu}$, a complex gravitino, a $U(1)_R$ gauge field A_μ, a scalar H and a conserved vector V_μ.
In ordinary (new-minimal) $3d$ supergravity A_μ, H and V_μ are auxiliary fields – determined by their equations of motion.
Here they are arbitrary background fields.

The relevant Chern-Simons contact terms are

- Flavor-R: $A \wedge da +$
- Lorentz: $\omega d\omega + \ldots$
- R-R: $AdA + \ldots$

Flavor-R Chern-Simons Term

The contact term in the two-point function of the R-current and a non-R flavor current is related by supersymmetry to a number of other contact terms. These are summarized by the supersymmetric local counter term

$$i\varepsilon^{\mu\nu\rho} a_\mu \partial_\nu \left(A_\rho - \frac{1}{2} V_\rho\right) + \frac{1}{4}\sigma R - DH + \cdots + \text{fermions}$$

This expression violates conformal invariance.

- The auxiliary fields $A_\mu - \frac{1}{2}V_\mu$ and H and the Ricci scalar R are not gauge invariant in conformal supergravity.
- These three fields couple to redundant operators in a CFT; e.g. R couples to T_μ^μ.

Lorentz Chern-Simons Term

The Lorentz Chern-Simons term can be supersymmetrized

$$\frac{i}{4}\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\omega_\mu\partial_\nu\omega_\rho+\frac{2}{3}\omega_\mu\omega_\nu\omega_\rho\right)+i\varepsilon^{\mu\nu\rho}\left(A_\mu-\frac{3}{2}V_\mu\right)\partial_\nu\left(A_\rho-\frac{3}{2}V_\rho\right)$$

$$+\cdots+\mathrm{fermions}$$

These terms are superconformal.

R-R Chern-Simons Term

There is another independent supersymmetric counter term

$$i\varepsilon^{\mu\nu\rho}\big(A_\mu - \frac{1}{2}V_\mu\big)\partial_\nu\big(A_\rho - \frac{1}{2}V_\rho\big) + \frac{1}{2}HR + \cdots + \text{fermions}$$

This expression violates conformal invariance.

- The auxiliary fields $A_\mu - \frac{1}{2}V_\mu$ and H and the Ricci scalar R are not gauge invariant in conformal supergravity.
- These three fields couple to redundant operators in a CFT; e.g. R couples to T^μ_μ.

A New Anomaly

The flavor-flavor and the Lorentz Chern-Simons terms can be completed to superconformal expressions.
Therefore, the corresponding contact terms can be nonzero in a superconformal theory.
As above, their fractional parts are physical.

The flavor-R and the R-R Chern-Simons terms cannot be completed to superconformal expressions. (The expressions above are supersymmetric but not conformal.)

What should we do about these nonconformal terms?

Given a superconformal field theory we would like to impose

- Supersymmetry
- Conformal symmetry
- All flavor and R-symmetries are compact

However, if the flavor-R and the R-R Chern-Simons contact terms are not integers, we cannot satisfy these requirements.

The most conservative approach is to cancel the contact terms by adding Chern-Simons terms with fractional coefficients.

Then, if the $U(1)_R$ gauge symmetry of the background fields is compact, the functional integral is not gauge invariant – it changes by a phase.

This is similar to the known framing anomaly [Witten]. Here it arises as a clash between large gauge transformations and superconformal symmetry.

Example: $\mathcal{N}=2$ SQED

This is a $U(1)$ gauge theory with N_f flavors of chiral superfields Q with charges $+1$ and $\widetilde{Q}$ with charge -1.

The theory is characterized by a gauge coupling constant e and the coefficient k of the $U(1)$ Chern-Simons term.

The theory has an $SU(N_f) \times SU(N_f) \times U(1)$ flavor symmetry and a $U(1)_R$ symmetry.
(In addition, it is invariant under charge conjugation, and it has a topological conserved current $\varepsilon_{\mu\nu\rho}F^{\nu\rho}$.)

The theory is free in the UV.
Below a crossover scale $M \sim ke^2$ it flows to a nontrivial superconformal field theory.

Chern-Simons Contact Terms in SQED

Explicit perturbative computations of various two-point functions uncover nonzero Chern-Simons contact terms.

- The two-point function of two $U(1)$ flavor currents leads to $\kappa^{ff} = \frac{\pi^2 N_f}{4k} + \mathcal{O}(\frac{1}{k^3})$.
- The two-point function of a $U(1)$ flavor current and a $U(1)_R$ current leads to $\kappa^{fR} = -\frac{N_f}{2k} + \mathcal{O}(\frac{1}{k^3})$.
- We did not compute explicitly the coefficients of the R-R and the Lorentz terms, but we expect them to be nonzero.

The nonzero values of the flavor-R and the R-R contact terms violate the superconformal symmetry of the IR theory.
We can cancel them by adding appropriate counter terms violating invariance under large gauge transformations of the background fields.

Placing an $\mathcal{N} = 2$ Theory on S^3

An $\mathcal{N} = 2$ theory with a $U(1)_R$ symmetry can be placed on S^3, while preserving supersymmetry [D. Sen; Rolmersberger; Kapustin, Willett, Yaakov].

We follow [Festuccia, NS] and turn on background superfields:

- We need to turn on $H = -\frac{i}{r}$. Here H is a scalar background superpartner of the metric and r is the radius.
- For any non-R $U(1)$ symmetry we can add a background gauge superfield with $D = \frac{i\sigma}{r}$ with complex σ.
- $\mathrm{Re}(\sigma) =$ real mass term. $\mathrm{Im}(\sigma)$ represents a choice of an R-symmetry [Jafferis; Hama, Hosomichi, Lee].
- The background fields H, σ and D do not have their standard reality – a possible problem with unitarity.

The Sphere Partition Function

- The sphere partition function $Z = e^{-F}$ of a unitary field theory should be real (even if it is not parity invariant). This follows from reflection positivity.
- If we want to preserve supersymmetry, we need to turn on imaginary H, which violates reflection positivity. Hence, the partition function could be complex.
- This should not be an issue in a superconformal theory, where H decouples.
- Nevertheless, explicit calculations based on localization exhibit complex answers for SCFTs on S^3.

The Phase of the Sphere Partition function

Starting in flat space, we find the four supersymmetric Chern-Simons contact terms. Two of them are not superconformal.

If we do not add bare counter terms to remove them, the superconformal field theory has nonconformal contact terms.

Substituting the complex background values of H, σ and D in these terms we find a nontrivial phase of Z (violating reflection positivity). It agrees with the explicit localization computations.

The anomaly discussed above is seen now as a clash between unitarity and full background gauge invariance.

F-maximization

Explore the sphere partition function $Z = e^{-F}$ as a function of $t = \mathrm{Im}(\sigma r)$, which represents the choice of R-symmetry.

We add local counter terms to restore the superconformal symmetry and unitarity.

- They are incompatible with the full background gauge invariance.
- They affect only $\mathrm{Im}(F)$.

Then,

- Vanishing of the one point function leads to $\partial_t \mathrm{Re} F = 0$ (conjectured by [Jafferis]).
- The coefficient τ of the flat space two-point function determines $\partial_t^2 \mathrm{Re} F = -\frac{\pi^2}{2}\tau > 0$. Hence, F is at a maximum (conjectured by [Jafferis, Klebanov, Pufu, Safdi]).
- The (flat space) observables $\kappa \bmod(1)$ and τ are calculable using localization on the sphere.
- They are independent of superpotential couplings including exactly marginal deformations.
- F-maximization is closely related to the "F-theorem."

Conclusions

- Contact terms are usually arbitrary.
- Chern-Simons contact terms lead to new computable observables.
- The natural way to describe them is in terms of counter terms of background gauge and gravity superfields.
- Some Chern-Simons contact terms are not superconformal – like an anomaly.
- In order to preserve supersymmetry on curved space, we should turn on various supergravity fields.
- The non-conformal Chern-Simons terms lead to a phase of the partition function (violation of unitarity).
- Removing these terms (by sacrificing invariance under large background gauge transformations) we prove the conjectured F-maximization.