Spectral Networks

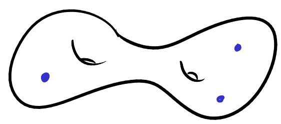
[w/ D. Gaiotto, G. Moore]

Aim: to describe some new geometric objects — networks of curves drawn on a punctured Riemann surface C — and explain how they control the BPS spectrum of $\mathcal{N}=2$ theories $S[\mathfrak{g}=A_{K-1}, C, D]$

The result will look like s.t. that could come from string theory — but in fact it's obtained by purely field theory considerations (w/ optimistic assumptions)

Players:

Riemann surface C



$K \geq 2$

Marked points z_ℓ $\quad 1 \leq \ell \leq n$

Parameters $m_\ell^{(i)} \in \mathbb{C}$ $\quad 1 \leq \ell \leq n,\ 1 \leq i \leq K$

(or more general "defects")

$\longrightarrow$ $\mathcal{N}=2$ SUSY QFT in $d=4$ [Witten, GMN, Gaiotto]

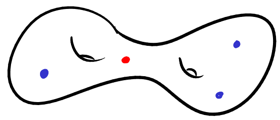
Tuple $\vec{\phi} = (\phi_2, \dots, \phi_K)$

ϕ_r a meromorphic r-differential on C with poles only at the z_ℓ, residues det. by $m_\ell^{(i)}$

$\longrightarrow$ a point of the Coulomb branch

IR physics: governed by the curve $\Sigma = \{ (\lambda, z) \in T^*C : \lambda^K + \sum_{r=2}^{K} \lambda^{K-r} \phi_r = 0 \}$

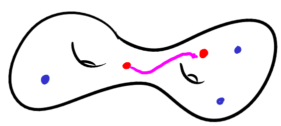
Point $z \in C$



$\longrightarrow$

Path P from z_1 to z_2

Phase ϑ



$\longrightarrow$

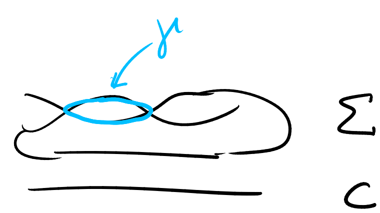
[AGGTV, Gaiotto]

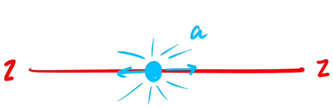
Now we may consider various sorts of BPS states. In particular:

4d BPS states: charge $\gamma \in H_1(\Sigma, \mathbb{Z})$

$Z = \oint_\gamma \lambda$

count $\Omega(\gamma, \vec{\varphi})$

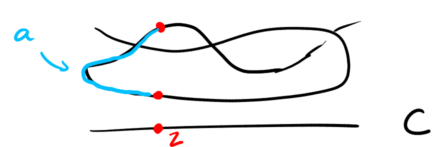


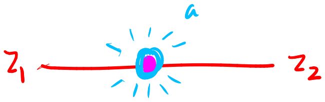


2d BPS states: charge $a \in$ - - -

$Z = \int_a \lambda$

count $\mu(a, z, \vec{\varphi})$

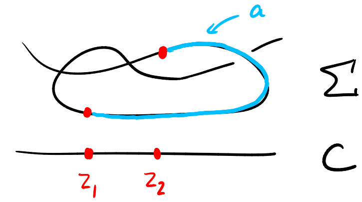




2d framed BPS states

$$Z = \int_a \lambda$$

count $\underline{\Omega}(a, P, \vartheta, z, \vec{\varphi})$



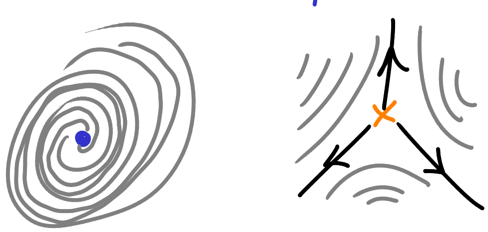
Now, how to compute the BPS spectra? By looking at their wall-crossing behavior + using just homotopy invariance, find that they are all governed by a single object:

spectral network $W(\vartheta, \vec{\varphi})$ drawn on C.

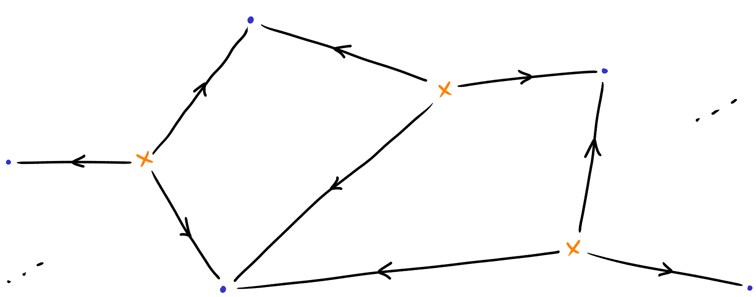
Begin with the special case $K=2$. So we have a curve C with a meromorphic quadratic differential ϕ_2. Assume ϕ_2 has only simple zeroes.

Given any $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ define ϑ-trajectories of ϕ_2 to be paths on C along which $e^{-i\vartheta}\sqrt{\phi_2}$ is real.

ϑ-trajectories make a singular foliation $F(\vartheta, \phi_2)$ of C:
$F(\vartheta, \phi_2)$ is singular at poles and zeroes of ϕ_2.



Focus on the 3 trajectories emanating from each zero; let $W(\vartheta, \phi_2)$ be their union.



Then:

- The framed BPS counts $\overline{\Omega}(a, P, \vartheta, \varphi_2)$ are determined by the <u>intersections</u> between P and $W(\vartheta, \varphi_2)$. [<u>Path lifting rule</u>] ($\Longrightarrow$ reps of line ops on $\mathbb{R}^3 \times S^1$)

- When ϑ is varied, $W(\vartheta, \varphi_2)$ sometimes <u>jumps</u>. $\Longrightarrow$ framed BPS states appear/disappear.
  How? By absorbing/emitting 4d BPS states! So:

$$\Omega(\gamma, \varphi_2) \in \mathbb{Z} \text{ count the } \underline{\text{jumps}} \text{ of } W(\vartheta, \varphi_2).$$

Concretely these jumps come from special trajectories which appear in $W(\vartheta, \varphi_2)$ at special ϑ:

$\Omega(\gamma) = 1$ "saddle connection"

$\Omega(\gamma) = -2$ "closed trajectory"

[Klemm-Lerche-Mayr-Vafa-Warner]
[Shapere-Vafa]

- An advantage of this point of view:
  Varying along different paths in $(\vartheta, \overline{\varphi})$ parameter space, framed BPS states appear/disappear in different order. But the end result must be path independent!
  $\Longrightarrow$ <u>Kontsevich-Soibelman wall-crossing formula</u> for $\Omega(\gamma, \varphi_2)$.

Now the general case, $K \geq 2$:

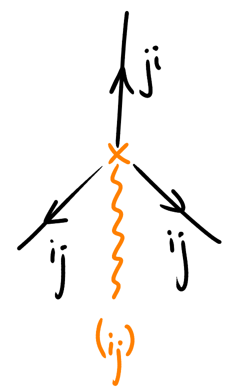
Locally, label the sheets of Σ by $i = 1, \dots, K$

Then we get K locally defined hol 1-forms $\lambda^{(i)}$ on C.

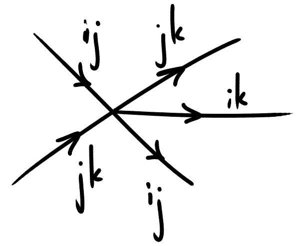
Define an ij-trajectory with phase ϑ to be a real curve on C with

$e^{-i\vartheta}(\lambda^{(i)} - \lambda^{(j)})$ real, positive.

Each branch point of type (ij) has 3 outgoing ij- or ji- trajectories.

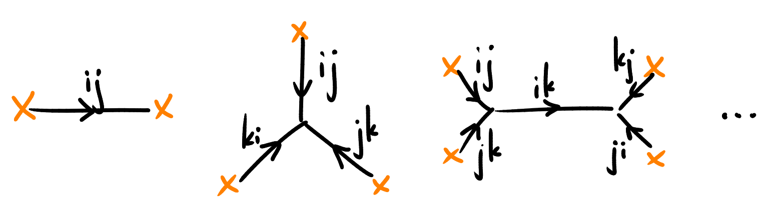


To define spectral network $W(\vartheta, \varphi)$: take all these trajectories plus extra ones generated at intersections:



As in $K=2$ case, this object determines all BPS counts!

In particular, its jumps as ϑ varies determine $\Omega(\gamma, \vec{\varphi})$. This is counting various kinds of "finite webs":



As before, these $\Omega(\gamma, \vec{\varphi})$ obey KSWCF as $\vec{\varphi}$ varies.

Vevs of line operators $\Rightarrow$ generalizn of FG coordinates assoc'd to <u>triangulations</u>