Recent Developments & Problems in G_2 Manifolds

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Notes: YHH

Work of $\underbrace{\text{D. Joyce ; A. Kovalev}}_{\text{construction of manifolds}}$; $\underbrace{\text{T. Walpuski; H. Sa tamp.}}_{\text{construction YM fields}}$

- G_2 manifolds : 7-d

def: 3-form φ ; $v_1, v_2 \in TM$

$$(v_1 \lrcorner \varphi) \wedge (v_2 \lrcorner \varphi) \wedge \varphi$$

is a volume form $\longrightarrow$ Riemannian metric

$*\varphi$.

Then : $d\varphi = d*\varphi = 0$ is G_2 structure

$$\Rightarrow \mathrm{Hol}(\text{metric}) \subset G_2 \subset SO(7)$$

Models:
Consider : $SU(3) < G_2$; $\mathbb{C}^3 \times \mathbb{R}_t$

(1) Let Z be Calabi-Yau (Θ, ω)

$$\varphi = \mathrm{Re}(\Theta) + \omega \wedge dt$$

$\therefore Z \times \mathbb{R}$ is G_2

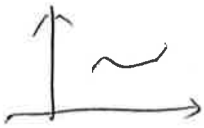
(2) $\mathbb{R}^4 \times \Lambda^2_+(\mathbb{R})$ $y_{I,J,K}$

$\omega_{I,J,K}$ $\varphi = \omega_I \wedge dy_I + \cdots + dy_I dy_J dy_K$

(3) $\mathbb{R}^3 \times S(\mathbb{R}^3)$

- They are all Ricci-flat manifolds. (2)
- M compact $\rightsquigarrow$ moduli of G_2-structures

$$H^3(M) \times H^4(M) \quad \rightsquigarrow \text{ graph}$$



- Q: Any compact examples? $\rightarrow$ Joyce.

Glueing constructions: start with asymptotic approx solution & deform.

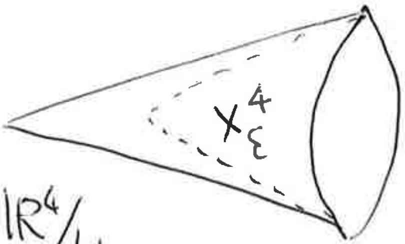
start. T^7 finite group Γ $\Rightarrow$ T^7/Γ singular G_2

choose Γ s.t. singularities are $T^3 \times \mathbb{R}^4/H$

$H \subset SO(4)$

take: X^4 ALE space ($\mathrm{Hol} = Sp(1)$), asymptotic to $\mathbb{R}^4/H$

$\underbrace{T^3 \times X^4_\varepsilon}_{\text{has } G_2 \text{ structure}}$



$\mathbb{R}^4/H$

- Kovalev construction: another example of compact G_2

W fano 3-fold. e.g. $\mathbb{CP}^3$.

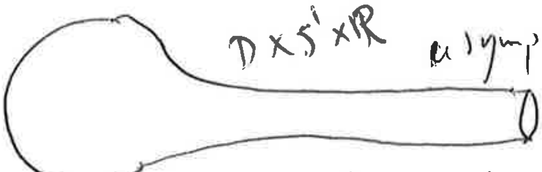
divisor $D \subset W$ in class $[-K_W]$

D is always a $K3$ surface w/ CY metric.

$\widetilde{W}$ blowup. $D \subset \widetilde{W}$ has trivial normal bundle

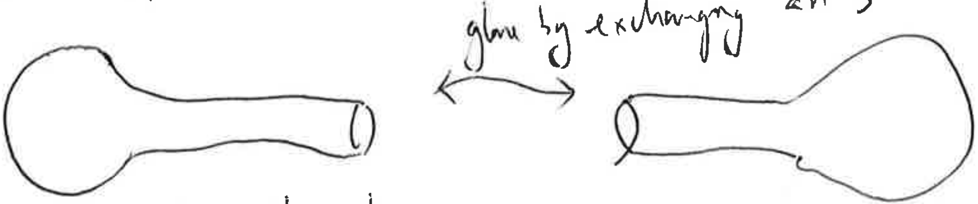
Then, $c_1(\widetilde{W} \setminus D) = 0$

$D \times S^1 \times \mathbb{R}$ asymptotic to cylinder.



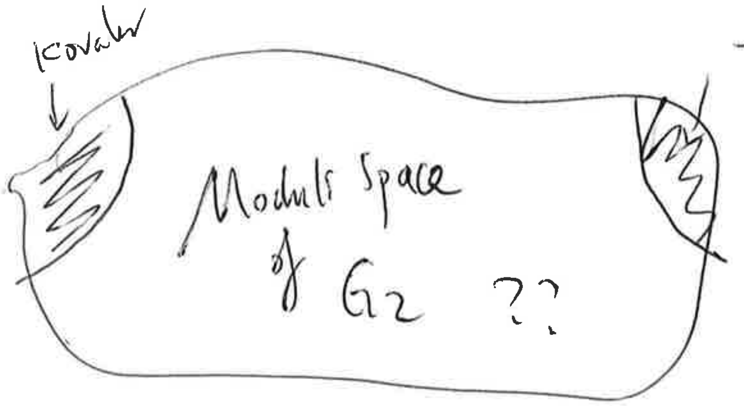
Tian-Yau metric, $\mathrm{Hol} = su(3)$

Take W_1 & W_2



$W_1 \times S^1 \times S^1 \times \mathbb{R}$ $\qquad$ $W_2 \times S^1 \times S^1 \times \mathbb{R}$

$$(D_1, I_{\text{complex str}}) \simeq (D_2, J_{\text{complex str}})$$



Cheeger – Tian :

Gauge Theory side of Story :

E bundle, w/ connection A
& curvature $F(A)$

$\downarrow$

M^7

$$\boxed{F(A) \wedge *\varphi = 0}$$

G_2-instanton equation

This is an elliptic equation

$$\Omega^0 \xrightarrow{d_A} \Omega^1 \xrightarrow{*\varphi \wedge d} \Omega^6 \xrightarrow{d} \Omega^7$$ seems over-determined.

gauge transf. A $\quad$ $F \wedge *\varphi$ $\quad$ $d(F\wedge *\varphi)$ But: for any A : $d_A(F \wedge *\varphi) = 0$

gives an elliptic complex

Energy identity : $\int \mathrm{Tr}(F^2) \wedge \varphi \underset{\text{chern-Weil}}{=} c\, p_1(E) \wedge [\varphi]$

"$\|F\|^2$"

everytime we have elliptic complex +
Energy identity.

We should ~~think~~ of "enumerative invariants".

(A rare structure → gives something rich)

- examples of solutions: Holomorphic bundle $\begin{matrix}\mathcal{E}\\ \downarrow \\ W\end{matrix}$. Fano

Suppose $\mathcal{E}|_D$ is stable $\Rightarrow$ $\mathcal{E}|_D$ admits HYM metric connection

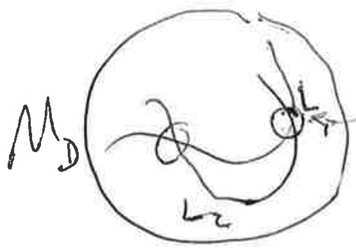
$$(F^{02} = 0 \; ; \quad F \cdot \omega = 0)$$

$\tilde{W} \setminus D$

[Sa-Earp] : Connection on $\begin{matrix}\mathcal{E}\\ \downarrow \\ \tilde{W} \setminus D\end{matrix}$

$\mathcal{M}_D$: moduli space of HYM connections over D

~~then~~



$\mathcal{M}_D$

L_1, L_2

L_1, L_2 Lagrangian submfd. of $\mathcal{M}_D$

then the solution should be in the intersections

- Extend to Joyce's construction: $T^3 \times X^4$, T^7/Γ

flat conn. described by rep. of certain gp.

moduli space of inst. on X

$$s : T^3 \to \mathcal{M}_X$$

<u>Fueter eq</u>: $$I \nabla_1 s + J \nabla_2 s + K \nabla_3 s = 0.$$