

Null Polygonal Wilson Loops in Full $\mathcal{N} = 4$ Superspace

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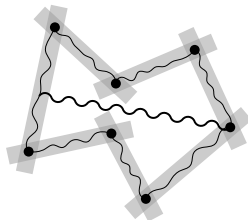
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arxiv:1203.1443

ETH

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich



with
S. He,
B. Schwab,
C. Vergu

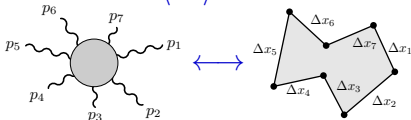
I. Introduction

Amplitudes vs. Null Polygonal Wilson Loops

Duality in Planar $\mathcal{N} = 4$ SYM:

- Colour-ordered scattering amplitudes \mathcal{A} .
- Null polygonal Wilson loop expectation value $\langle \mathcal{W} \rangle$.

$$\mathcal{A}_{\text{MHV}} = \mathcal{A}_{\text{MHV}}^{\text{tree}} \langle \mathcal{W}_{\text{bosonic}} \rangle.$$



[Alday, Maldacena] [Drummond, Korchemsky, Sokatchev] [Brandhuber, Heslop, Travaglini]

Identify:

- Null (on-shell) momenta $p \leftrightarrow$ null segments Δx .
- Momentum conservation \leftrightarrow closure of Wilson loop.

Obey conformal + dual conf. = Yangian symmetry.

[Drummond, Henn, Korchemsky, Sokatchev] [Drummond, Henn, Plefka]

What about Supersymmetry?

$$\mathcal{A} = \mathcal{A}_{\text{MHV}}^{\text{tree}} \langle \mathcal{W}_{\text{chiral}} \rangle.$$

$\langle \mathcal{W}_{\text{chiral}} \rangle$ has half of super (conformal) symmetry Q, \bar{S} , but not \bar{Q}, S .

Would like to use symmetries to construct \mathcal{A} , $\langle \mathcal{W} \rangle$ exactly.

Need to understand symmetries and anomalies.

[Mason, Skinner] [Caron-Huot, 1010.1167]

Fully Supersymmetric Wilson Loops

Transformation of chiral WL under \bar{Q} computed.

Use anomaly to reconstruct chiral WL expectation value.

[Caron-Huot]
1105.5606

[Caron-Huot, He]
1112.1060

Need Wilson loop \mathcal{W} in full superspace for full supersymmetry.

- How to define null polygons in full superspace?
- What is the SYM connection in full superspace?
- How to compute the expectation value?
- How is the full Wilson loop related to amplitudes?

This Talk: Want to compute WL expectation value in full superspace.

- Can compute from component fields. Better not!
Mess: e.g. compare components and superspace for $A_{\text{MHV}}^{\text{tree}}$.
- No fully covariant off-shell superspace for $\mathcal{N} = 4$ SYM.
- Nevermind that. Use $\mathcal{N} = 4$ superspace anyway!

Outline

$\mathcal{N} = 4$ Super Yang–Mills in Full Superspace

- Define gauge theory on $\mathcal{N} = 4$ superspace. Constraints.
- Quantise and derive two-point functions.

Wilson Loops on Null Polygons in Superspace

- How to define null polygons?
- How to compute Wilson loop efficiently?

Twistors

- Simplify expressions using twistor variables.
- Take seriously: Wilson loop in ambitwistor space.

Regularisation

- UV divergences at cusps need to be regularised. How?

II. $\mathcal{N} = 4$ SYM in Superspace

$\mathcal{N} = 4$ Super Yang–Mills Formulations

Various formulations of $\mathcal{N} = 4$ Super Yang–Mills theory:

- YM + matter,
- $\mathcal{N} = 1$ SYM + matter,
- $\mathcal{N} = 2$ SYM + matter in harmonic superspace,
- 10D YM + fermion reduced to 4D,
- 10D YM + fermion reduced to $4 - 2\epsilon$ dimensions,
- $\mathcal{N} = 4$ SYM on the light cone.

All formulations have:

- certain benefits,
- certain drawbacks,
- several followers,
- many enemies.

Eventually, all produce the same results.

Clearly: None displays full $\mathcal{N} = 4$ superspace $(x, \theta^a, \bar{\theta}_a)$ structure.

Inconvenient for computing a Wilson loop in **full** superspace.

$\mathcal{N} = 4$ Superspace

4 real bosonic coordinates $(-+++)$, 8 complex fermionic coordinates

$$X = (x^{\beta\dot{\alpha}}, \theta^{\beta a}, \bar{\theta}_b \dot{\alpha}),$$

Our conventions:

- Use **spinor** notation: $x \sim \sigma_{\mu} x^{\mu}$.
 $x, \theta, \bar{\theta}$ are $2 \times 2, 2 \times 4, \bar{4} \times 2$ matrices, respectively.
- Assume **real** Minkowski superspace: $x^{\dagger} = x, \theta^{\dagger} = \bar{\theta}$.
No need for (2,2) signature or complex spacetime.

Superspace has translation-invariant **vielbein** $(e, d\theta, d\bar{\theta})$ with

$$e = dx - id\theta\bar{\theta} - i\theta d\bar{\theta}.$$

Superspace has non-trivial **torsion** ($\{Q, \bar{Q}\} \sim P$)

$$de = -2id\theta d\bar{\theta}.$$

Yang–Mills Theory on $\mathcal{N} = 4$ Superspace

Introduce a connection A on superspace

[Sohnius]

$$A = \text{Tr}(d\theta A_\theta) - \text{Tr}(d\bar{\theta} A_{\bar{\theta}}) + \text{Tr}(e A_x).$$

Connection A has (way) more component fields than $\mathcal{N} = 4$ SYM.
Force certain components of field strength $F = dA + A^2$ to vanish:

$$F = -\frac{1}{2} \text{Tr}(d\theta^\top \varepsilon d\theta \bar{\Phi}) + \text{Tr}(e^\top \varepsilon d\theta \bar{\Psi}) + \frac{1}{2} \text{Tr}(e^\top \varepsilon e \bar{\Gamma}) \\ - \frac{1}{2} \text{Tr}(d\bar{\theta} \varepsilon d\bar{\theta}^\top \Phi) + \text{Tr}(d\bar{\theta} \varepsilon e^\top \Psi) + \frac{1}{2} \text{Tr}(e \varepsilon e^\top \Gamma), \quad \Phi^{ab} = \frac{1}{2} \varepsilon^{abcd} \bar{\Phi}_{cd}.$$

Remaining superfields Φ, Ψ, Γ contain scalar, spinor and field strength.

Jacobi Identity $dF + [A, F] = 0$ implies:

- Supersymmetry relations between the fields.
- Equations of motion of classical $\mathcal{N} = 4$ SYM.

Inconvenient for full quantisation.

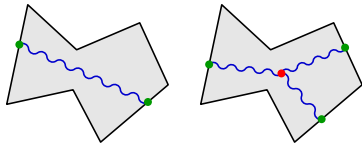
Wilson Line

Straight-forward to define **Wilson line** \mathcal{W} on full superspace

$$\mathcal{W}_\gamma = \text{P exp} \int_\gamma A.$$

Want to compute $\langle \mathcal{W}_\gamma \rangle$, need to quantise, requires:

- Feynman propagator Δ_F ,
- vertices (two loops and above),
- ghosts (two loops and above).



Start simple; content ourselves with one loop:

- Use canonical quantisation.
- Derive propagator $\Delta_F = \langle 0 | T[A A'] | 0 \rangle$
- Linearised/abelian theory sufficient.
- Can work on-shell.

Linearised $\mathcal{N} = 4$ SYM and Prepotentials

There is a simple ansatz to solve linearised E.O.M.:

$$A_{\theta, a\beta} = \varepsilon_{\beta\delta} D_{a\gamma} B^{\gamma\delta}, \quad A_{\bar{\theta}, \dot{\alpha}^b} = \varepsilon_{\dot{\alpha}\dot{\delta}} \bar{D}_{\dot{\gamma}^b} \bar{B}^{\dot{\gamma}\dot{\delta}}.$$

The prepotentials B, \bar{B}

- are (anti)-chiral: $\bar{D}B = 0, D\bar{B} = 0,$
- are harmonic: $D^2B = 0, \bar{D}^2\bar{B} = 0,$
- are self-conjugate: $D^2 \cdot B \sim \bar{D}^2 \cdot \bar{B},$
- can be axial gauge fixed: $B^{\alpha\gamma} = \rho^\alpha \rho^\gamma B, \bar{B}^{\dot{\alpha}\dot{\gamma}} = \bar{\rho}^{\dot{\alpha}} \bar{\rho}^{\dot{\gamma}} \bar{B}$ ($\rho, \bar{\rho}$ fixed).

Solution in on-shell momentum space (solves chiral harmonic)

$$B(x^+, \theta) = \int d^2\lambda d^2\bar{\lambda} d^{0|4}\eta \exp(i\langle\lambda|x^+|\bar{\lambda}\rangle + i\langle\lambda|\theta|\eta\rangle) \frac{B(\lambda, \bar{\lambda}, \eta)}{\langle\lambda|\rho\rangle^2}.$$

Complete solution: $B(\lambda, \bar{\lambda}, \eta)$ has 16 components for each $p = \lambda\bar{\lambda}$.

Two-Point Correlators

Need to find $\langle 0|B(\lambda, \bar{\lambda}, \eta) B(\lambda', \bar{\lambda}', \eta')|0\rangle$. Canonical quantisation?

[NB, He
Schwab
Vergu]

Action for $\mathcal{N} = 4$ superspace not known/straight-forward. Construct:

$$\langle 0|B B'|0\rangle \sim \oint \frac{dz}{z} \delta^2(\lambda' + z^{-1}\lambda) \delta^2(\bar{\lambda}' - z\bar{\lambda}) \delta^{0|4}(\eta' - z\eta) \theta(E(\lambda, \bar{\lambda})).$$

Mixed correlator via fermionic FT $\bar{B}(\bar{\eta}) \sim \int d^{0|4}\eta \exp(\eta\bar{\eta})B(\eta)$:

$$\langle 0|B \bar{B}'|0\rangle \sim \oint \frac{dz}{z} \delta^2(\lambda' + z^{-1}\lambda) \delta^2(\bar{\lambda}' - z\bar{\lambda}) \exp(\eta\bar{\eta}') \theta(E(\lambda, \bar{\lambda})).$$

Transform to position space

$$\langle 0|B B'|0\rangle \sim \frac{\delta^{0|4}(\langle \theta|x^+|\bar{\rho}\rangle)}{\langle \rho|x^+|\bar{\rho}\rangle^4 (x^+)^2} + \dots, \quad \langle 0|B \bar{B}'|0\rangle \sim \frac{x^2 \log(x^2)}{\langle \rho|x|\bar{\rho}\rangle^2} + \dots$$

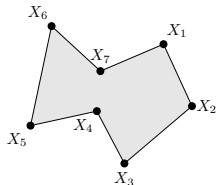
Can now compute one-loop Wilson loop expectation values.

III. Wilson Loop

Null Polygonal Wilson Loop

Consider a null polygon in full superspace.

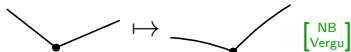
- Sequence of vertices $X_j = (x_j, \theta_j, \bar{\theta}_j)$.
- Neighbours are null separated:



$$(x_{j,j+1}^\pm)^2 = 0, \quad \langle x_{j,j+1}^+ | \theta_{j,j+1} = 0, \quad \bar{\theta}_{j,j+1} | x_{j,j+1}^- \rangle = 0.$$

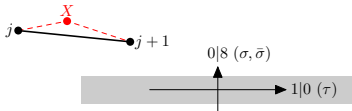
How to connect the vertices by a curve $X(\tau)$?

- “Straight” line $X(\tau) = X_0 + \tau \Delta X$ not stable under superconformal.
- Demand that $X(\tau)$ is everywhere null separated from vertices.



$$x = x_j + \tau \lambda_j \bar{\lambda}_j + i \lambda_j \sigma \bar{\theta}_j - i \theta_j \bar{\sigma} \bar{\lambda}_j,$$

$$\theta = \theta_j + \lambda_j \sigma, \quad \bar{\theta} = \bar{\theta}_j + \bar{\sigma} \bar{\lambda}_j.$$



Solution $X(\tau, \sigma, \bar{\sigma})$ depends on 8 free fermionic curve parameters $\sigma, \bar{\sigma}$.

Null segment is **fat**. Which curve $X(\tau)$ to pick?

- Connection A flat on $X(\tau, \sigma, \bar{\sigma})$: $F|_{\text{segment}} = 0$.
- All curves $X(\tau)$ on $X(\tau, \sigma, \bar{\sigma})$ physically equivalent!



[Witten]

Segment Potential Shift

Can compute WL expectation value by brute force.

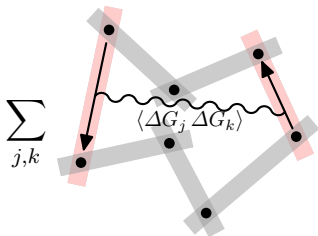
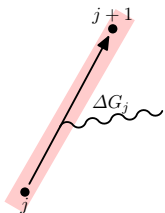
Better: Integrate connection $A = dG_j + d\bar{G}_j$ on fat segment j

[NB, He
Schwab
Vergu]

$$G_j(x^+, \theta) \sim \int d^2\lambda d^2\bar{\lambda} d^{0|4}\eta e^{i\langle\lambda|x^+|\bar{\lambda}\rangle + i\langle\lambda|\theta|\eta\rangle} \frac{\langle\lambda_j|\rho\rangle B(\lambda, \bar{\lambda}, \eta)}{\langle\lambda|\rho\rangle\langle\lambda|\lambda_j\rangle}.$$

(Abelian) Wilson loop \mathcal{W} is sum over potential shifts across segments

$$\log \mathcal{W} = \oint A = \sum_{j=1}^n \Delta G_j, \quad \Delta G_j = G_j(X_{j+1}) - G_j(X_j).$$

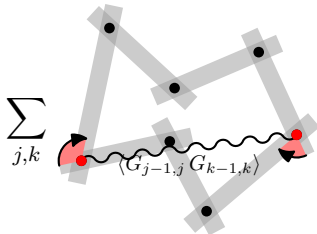
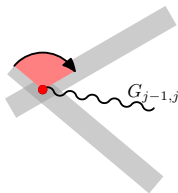


Vertex Potential Shift

Can rearrange Wilson loop as sum over vertices

[Mason] [NB, He]
[Skinner] [Schwab]
[Vergu]

$$\oint A = \sum_{j=1}^n G_{j-1,j}, \quad G_{j-1,j} = G_{j-1}(X_j) - G_j(X_j),$$



Vertex potential shift reads

$$G_{j-1,j} \sim \int d^2\lambda d^2\bar{\lambda} d^0|{}^4\eta e^{i\langle\lambda|x_j^+|\bar{\lambda}\rangle+i\langle\lambda|\theta_j|\eta\rangle} \frac{\langle\lambda_{j-1}|\lambda_j\rangle B(\lambda, \bar{\lambda}, \eta)}{\langle\lambda|\lambda_{j-1}\rangle\langle\lambda|\lambda_j\rangle}.$$

Localised at point X_j . Gauge dependence on ρ drops out!

Vertex Correlators: Purely Chiral

Compute correlator of vertex potential shifts $\langle 0 | G_{j-1,j} G_{k-1,k} | 0 \rangle$.

Chiral-chiral: Integrals over delta-functions

$$\sim \int d^2\lambda d^2\bar{\lambda} \theta(E) e^{i(\lambda|x_{j,k}^+|\bar{\lambda})} \frac{\delta^{0|4}(\langle\lambda|\theta_{j,k}\rangle\langle j-1|j\rangle\langle k-1|k\rangle)}{\langle\lambda|j-1\rangle\langle\lambda|j\rangle\langle\lambda|k-1\rangle\langle\lambda|k\rangle}.$$

Perform remaining Fourier integral (differential operators)

$$\sim \frac{\langle j-1|j\rangle\langle k-1|k\rangle\delta^{0|4}(\theta_{j,k}|x_{j,k}^+|\bar{\rho})}{\langle j-1|x_{j,k}^+|\bar{\rho}\rangle\langle j|x_{j,k}^+|\bar{\rho}\rangle\langle k-1|x_{j,k}^+|\bar{\rho}\rangle\langle k|x_{j,k}^+|\bar{\rho}\rangle(x_{j,k}^+)^2}.$$

- Well-known vertex correlator: R-invariant.
- Yields tree-level NMHV/MHV upon summation.
- Reference spinor $\bar{\rho}$ drops out in sum.
- Purely anti-chiral correlator is complex conjugate.

[Drummond
Henn]

Vertex Correlators: Mixed Chiral

Compute mixed chirality correlator $\langle 0|G_{j-1,j}\bar{G}_{k-1,k}|0\rangle$.

$$\sim \int d^2\lambda d^2\bar{\lambda} \theta(E) e^{i\langle\lambda|x_{j,k}^{+-}|\bar{\lambda}\rangle} \frac{\langle j-1|j\rangle[k-1|k]}{\langle\lambda|j-1\rangle\langle\lambda|j\rangle[\bar{\lambda}|k-1][\bar{\lambda}|k]}.$$

Perform remaining Fourier integral (differential operators)

$$\begin{aligned} &\sim \frac{1}{2} \log \frac{\langle j-1|x_{j,k}^{+-}|k\rangle \langle j|x_{j,k}^{+-}|k-1\rangle}{\langle j-1|x_{j,k}^{+-}|k-1\rangle \langle j|x_{j,k}^{+-}|k\rangle} \log \left(\langle j-1|x_{j,k}^{+-}|k-1\rangle \langle j|x_{j,k}^{+-}|k\rangle \right) \\ &\quad - \text{Li}_2 \frac{\langle j-1|x_{j,k}^{+-}|k\rangle \langle j|x_{j,k}^{+-}|k-1\rangle}{\langle j-1|x_{j,k}^{+-}|k-1\rangle \langle j|x_{j,k}^{+-}|k\rangle}. \end{aligned}$$

- Expected one-loop form: $\text{Li}_2 + \log^2$, superconformal cross ratios.
- Singular for $|k-j| \leq 2$; regularisation needed.
- Fermionic contribution only in $x_{jk}^{+-} := x_k^- - x_j^+ + 2i\theta_j\bar{\theta}_k$.
- Used same correlator as for purely chiral case up to fermionic FT.

VEV vs. Time-Ordered Expectation Value

In fact, computed $\langle 0|W|0\rangle$ instead of $\langle W\rangle = \langle 0|T[W]|0\rangle$.

Used $\Delta = \langle 0|\Phi\Phi'|0\rangle$ instead of $i\Delta_F = \langle\Phi\Phi'\rangle$.

- $\Delta \sim \theta(E)\delta(p^2)$ is on-shell, $i\Delta_F \sim 1/p^2$ is off-shell.
- E.O.M. required for exploiting constraints: flatness on segments.

Note that difference is in imaginary part (scalar field)

$$i\Delta_F - \Delta \sim i\theta(\pm t)\delta(x^2).$$

Difference $\langle W\rangle - \langle 0|W|0\rangle$ is small: lower transcendentality.

- Leading transcendentality in $\langle 0|W|0\rangle$ correct! Disregard lower ones.
- Compute cut $\text{disc}\langle W\rangle$, $\text{disc}i\Delta_F \sim \delta(p^2)$: on-shell!
- Brute force calculation in position space: $i\Delta_F = T[\Delta]$. Okay!

IV. Twistors

Fat Null Lines and Twistor Variables

Recall fat null line:

$$x = x_j + \tau \lambda_j \bar{\lambda}_j + i \lambda_j \sigma \bar{\theta}_j - i \theta_j \bar{\sigma} \bar{\lambda}_j, \quad \theta = \theta_j + \lambda_j \sigma, \quad \bar{\theta} = \bar{\theta}_j + \bar{\sigma} \bar{\lambda}_j.$$

Equivalent to solution of twistor equations

$$\begin{aligned} \langle \lambda_j | x^+ = \langle \lambda_j | x_j^+ &=: \mu_j, & x^- | \bar{\lambda}_j = x_j^- | \bar{\lambda}_j &=: \bar{\mu}_j, \\ \langle \lambda_j | \theta = \langle \lambda_j | \theta_j &=: \chi_j, & \bar{\theta} | \bar{\lambda}_j = \bar{\theta}_j | \bar{\lambda}_j &=: \bar{\chi}_j. \end{aligned}$$

The $4|4$ projective vectors $W_j := (\lambda_j, \mu_j, \chi_j)$ are (momentum) twistors. $\bar{W}_j := (\bar{\mu}_j, \bar{\lambda}_j, \bar{\chi}_j)$ are conjugate twistors; (W_j, \bar{W}_j) is real ambitwistor.

Twistor product is superconformal invariant

$$\langle j, k \rangle := W_j \cdot \bar{W}_k \sim \langle \lambda_j | x_{jk}^{+-} | \bar{\lambda}_k \rangle.$$

Note: $\langle j, k \rangle = 0$ for $k = j, j \pm 1$.

Correlators using Twistors

Purely chiral vertex correlator using twistors $W = (\omega, \chi)$

$$\frac{\delta^{0|4} (\chi_{\star} \varepsilon_{ABCD} \omega_{j-1}^A \omega_j^B \omega_{k-1}^C \omega_k^D + 4 \text{ perm.})}{\varepsilon_{ABCD} \omega_{j-1}^A \omega_j^B \omega_{k-1}^C \omega_k^D \times 4 \text{ perm.}}$$

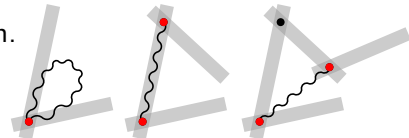
- Superconformal invariant. Reminiscent of superdeterminant.
- Reference twistor W_{\star} disappears in sum.

Mixed chiral vertex correlator using twistors

$$\frac{1}{2} \log \frac{\langle j-1, k \rangle \langle j, k-1 \rangle}{\langle j-1, k-1 \rangle \langle j, k \rangle} \log (\langle j-1, k-1 \rangle \langle j, k \rangle) - \text{Li}_2 \frac{\langle j-1, k \rangle \langle j, k-1 \rangle}{\langle j-1, k-1 \rangle \langle j, k \rangle}.$$

- Only superconformal twistor brackets.
- Weight in W_j and \bar{W}_j vanishes in sum.
- Singular when one $\langle j, k \rangle = 0$.

Regularise three terms:



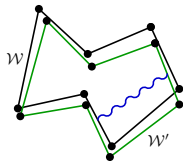
V. Regularisations

Framing

“Point-splitting”: Correlator of two nearby Wilson loops $\mathcal{W}, \mathcal{W}'$

[Bullimore
Skinner]

$$\langle \mathcal{W} \rangle_{\text{frame}} = \left(\frac{\langle \mathcal{W} \mathcal{W}' \rangle}{\langle \mathcal{W} \rangle \langle \mathcal{W}' \rangle} \right)^{1/2}$$



Shift second polygon using twistor variables (weights preserved)

[NB, He
Schwab
Vergu]

$$W'_j = W_j + \epsilon \frac{\langle W_j, \bar{W}_* \rangle}{\langle W_*, \bar{W}_* \rangle} W_* + \mathcal{O}(\epsilon^2).$$

Twistor products regularised: $\mathcal{O}(\epsilon)$ instead of 0 for $|j - k| \leq 1$

$$\langle j, k' \rangle = \langle j, k \rangle + \epsilon \frac{\langle j, \bar{*} \rangle \langle *, k \rangle}{\langle *, \bar{*} \rangle} + \mathcal{O}(\epsilon^2).$$

Result depends on reference twistor W_*, \bar{W}_* : Axial symmetry breaking.
Result diverges for $\epsilon \rightarrow 0$ as it should (UV).

Supersymmetric Extrapolation

Superspace formalism tied to 4D. Cannot do dimensional regularisation.

Construct a supersymmetric expression:

[NB, He
Schwab
Vergu]

- Start with known bosonic result.
- Express through $\langle j|k\rangle$, $[j|k]$, $\langle j|x_{jk}|k\rangle$.
- Adjust proper weights in spinors $\langle j|$ and $|k\rangle$.
- Lift $\langle j|x_{jk}|k\rangle$ to $\langle j|x_{jk}^{+-}|k\rangle$ and thus $\langle j, k\rangle$.
- Obtain an expression in $\langle j|k\rangle$, $[j|k]$ and $\langle j, k\rangle$.

Features:

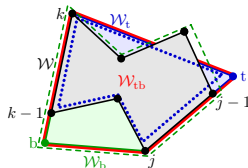
- $\langle j, k\rangle$ is superconformal invariant.
- $\langle j|k\rangle$, $[j|k]$ break (super)conformal boosts.
- Expression has correct collinear behaviour.
- First principles derivation desirable.

Boxed Wilson Loop

Compute a UV-finite quantity instead. Boxed Wilson Loop:

[Alday, Gaiotto
Maldacena
Sever, Vieira]

$$C_{\text{box}} = \frac{\langle \mathcal{W} \rangle \langle \mathcal{W}_{tb} \rangle}{\langle \mathcal{W}_t \rangle \langle \mathcal{W}_b \rangle}.$$



Features:

- Result is finite and independent of regularisation.
- Needs only minor regularisation for special vertices.
- Only twistor brackets $\langle j, k \rangle$: Manifestly superconformal!
- Framed and guessed results reduce to this.

Test of collinear behaviour. Consistent!

[NB, He
Schwab
Vergu]

VI. Conclusions

Conclusions

$\mathcal{N} = 4$ Super Yang–Mills in Full Superspace

- Can indeed be used in calculations.

Null Polygonal Wilson Loops in Full Superspace

- Null polygons are fat.
- Computed one-loop expectation value efficiently.
- Contains NMHV tree and fully supersymmetric MHV loop.
- Dual description in ambitwistor space: Null lines.

Regularisation and Symmetries

- Can regularise in several ways.
- Superconformal symmetry broken in different ways.
- Can compute finite quantities with superconformal symmetry.